# ON RIEMANN'S EXPLICIT FORMULA FOR THE NUMBER OF PRIMES

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Abstract. In this paper, we will discuss Riemann's explicit formula for the number of primes less than or equal to a real number $x$, often denoted as $\pi(x)$. We give the proof of this remarkable formula and will explore its implications assuming that the Riemann hypothesis holds.

## 1. Background information

We will need a few important functions and theorems before we present the proof of the explicit formula. First, we define the $\pi$ function:

**Definition 1.1.** Let $g(x)$ be the number of primes less than or equal to a real number $x$. Then we define

$$\pi(x) = \frac{1}{2}\lim_{h\to 0}(g(x+h) - g(x-h));$$

in other words, $\pi(x) = g(x)$ except for when $x$ is a prime number, in which case $\pi(x) = g(x) - 0.5$.

Now we define the $J$ function:

**Definition 1.2.** We define the function $J(x)$ as follows:

$$J(x) = \sum_{k=1}^{\infty} \frac{\pi(x^{1/k})}{k}.$$

This is the function that we will explicitly describe in this paper. To see how $J(x)$ relates to $\pi(x)$, one can derive the following using Mobius inversion:

**Theorem 1.3.**

$$\pi(x) = \sum_{k=1}^{\infty} \frac{\mu(k)J(x^{1/k})}{k}.$$

*Date*: June 10, 2024.

*Proof.* This isn't too difficult to prove, but for completeness we provide the proof. Plug in the definition of $J(x)$ as given by Definition 1.2 to get

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{\mu(k)J(x^{1/k})}{k} &= \sum_{k=1}^{\infty}\frac{\mu(k)}{k}\sum_{m=1}^{\infty}\frac{\pi(x^{1/mk})}{m}\\ &= \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{\pi(x^{1/mk})\mu(k)}{mk}\\ &= \sum_{n=1}^{\infty}\frac{\pi(x^{1/n})}{n}\sum_{d\mid n}\mu(d)\\ &= \pi(x)\end{aligned}$$

by changing the order of summation and using the fact that $\sum_{d|n}\mu(d) = 0$ for $n > 1$, as desired. ■

Riemann's explicit's formula then states the following:

**Theorem 1.4** (Riemann's Explicit Formula)**.** *We have that*

$$J(x) = \mathrm{Li}(x) - \sum_{\rho}(\mathrm{Li}(x^{\rho}) + \mathrm{Li}(x^{1-\rho})) + \int_x^{\infty}\frac{dt}{t(t^2-1)\log t} - \log 2,$$

*where the sum is taken over all nontrivial zeroes of the Riemann zeta function, arranged in increasing magnitude of imaginary part.*

In this paper, we will prove this remarkable formula.

## 2. $\xi$ and its properties

To prove the explicit formula, we need to define a new function called $\xi$.

**Definition 2.1.** We define the function $\xi$ as

$$\xi(s) = \frac{1}{2}\pi^{-s/2}s(s-1)\Gamma(s/2)\zeta(s).$$

The reason that this function comes into play is that it is symmetric across 1:

**Proposition 2.2.** *We have* $\xi(s) = \xi(1-s)$.

*Proof.* We can plug in the definition of $\xi$ and turn this into

$$\frac{1}{2}\pi^{-s/2}s(s-1)\Gamma(s/2)\zeta(s) = -\frac{1}{2}\pi^{(s-1)/2}s(s-1)\Gamma((1-s)/2)\zeta(1-s).$$

It is known that $\zeta$ satisfies the functional equation $\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$, so plugging that in gives

$$\frac{1}{2}\pi^{-s/2}s(s-1)\Gamma(s/2)2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) = \frac{1}{2}\pi^{(s-1)/2}s(s-1)\Gamma((1-s)/2)\zeta(1-s).$$

Dividing both sides by $\frac{1}{2}s(s-1)\zeta(1-s)$ and bringing the factors of $\pi$ together on the left-hand side, we get

$$\pi^{s/2-1}\Gamma(s/2)2^s\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) = \pi^{s/2-1/2}\Gamma((1-s)/2).$$

Dividing both sides by $\pi^{s/2-1/2}$, it suffices to show that

$$2^s \pi^{-1/2} \Gamma(s/2)\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right) = \Gamma((1-s)/2).$$

We now recall Legendre's Gamma duplication formula, which states that

$$\Gamma(2t) = \frac{2^{2t-1}}{\sqrt{\pi}}\Gamma(t)\Gamma\left(t+\frac{1}{2}\right).$$

Setting $t = \frac{1-s}{2}$ tells us that

$$\Gamma(1-s) = \frac{2^{-s}}{\sqrt{\pi}}\Gamma((1-s)/2)\Gamma(1-s/2).$$

Plugging this in, we get

$$2^s \pi^{-1/2} \Gamma(s/2) \cdot \frac{2^{-s}}{\sqrt{\pi}}\Gamma((1-s)/2)\Gamma(1-s/2)\sin\left(\frac{\pi s}{2}\right) = \Gamma((1-s)/2).$$

Simplifying the left-hand side and dividing both sides by $\frac{\Gamma((1-s)/2)}{\pi}$, we get

$$\Gamma(s/2)\Gamma(1-s/2)\sin\left(\frac{\pi s}{2}\right) = \pi.$$

Dividing both sides by $\sin\left(\frac{\pi s}{2}\right)$, we get

$$\Gamma(s/2)\Gamma(1-s/2) = \frac{\pi}{\sin\left(\frac{\pi s}{2}\right)},$$

which is precisely the Gamma reflection formula. Since all of our steps are reversible, we conclude that $\xi(s) = \xi(1-s)$. ■

The next step is to classify the zeroes of $\xi$:

**Theorem 2.3.** *The zeroes of $\xi$ are precisely the nontrivial zeroes of $\zeta$.*

*Proof.* Suppose that $s \in \mathbb{C}$ satisfies $\xi(s) = 0$. Then at least one of the factors in the expression given in Definition 1.4 must equal 0. There are 3 cases:

**Case 1:** $s = 0$ or $s = 1$. Then $\zeta(s)$ diverges and outgrows $s(s-1)$, so there are no zeroes in this case.

**Case 2:** $\zeta(s) = 0$ and $s$ is a trivial zero of $\zeta$. Then $s$ is a negative even integer, so $\Gamma(s/2)$ diverges and also outgrows $\zeta(s)$. Thus there are no zeroes in this case.

**Case 3:** $\zeta(s) = 0$ and $s$ is a nontrivial zero of $\zeta$. Then $s$ is known to have real part between 0 and 1, exclusive, so $\Gamma(s/2)$ doesn't diverge. Neither do any of the other terms, so in this case we have zeroes.

Therefore, the only zeroes of $\xi$ are the nontrivial zeroes of $\zeta$, as desired. ■

We now quote the following theorem from Hadamard:

**Theorem 2.4** (Hadamard)**.** *We have that*

$$\xi(s) = \xi(0)\prod_{\xi(\rho)=0}\left(1-\frac{s}{\rho}\right).$$

We make no effort to prove this theorem in our paper, but what this theorem says is that we can "factor" $\xi$ into (infinitely many) linear factors, each with zero equal to some zero of $\xi$, and therefore express $\xi$ as a polynomial in $s$. Ultimately, this is how we will connect $\xi$ to the nontrivial zeroes of $\zeta$.

## 3. Connecting $J$ with $\zeta$

Our next objective is to relate the $J$ function defined in Section 1 with $\zeta$. The heart of this section is the following theorem:

**Theorem 3.1.** *For all $s \in \mathbb{C}$ with real part greater than* 1*, we have that*

$$\log \zeta(s) = s \int_0^\infty J(x) x^{-s-1} dx.$$

*Proof.* The Euler product for $\zeta$ tells us that

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

Taking the natural logarithm of both sides and using the Taylor series expansion for log, we get

$$\log \zeta(s) = \sum_{p \text{ prime}} \sum_{n=1}^\infty \frac{p^{-ns}}{n}.$$

Swapping the order of summation, we get

$$\log \zeta(s) = \sum_{n=1}^\infty \sum_{p \text{ prime}} \frac{p^{-ns}}{n}$$

One can verify that $s \int_{p^n}^\infty x^{-s-1} dx = p^{-ns}$ (this is just standard calculus). Plugging this in, we get

$$\log \zeta(s) = s \sum_{n=1}^\infty \sum_{p \text{ prime}} \frac{1}{n} \int_{p^n}^\infty x^{-s-1} dx.$$

Bringing the summations inside the integral via summation swapping again, we get

$$\log \zeta(s) = s \int_0^\infty \sum_{n=1}^\infty \sum_{p^n \leq x} \frac{1}{n} \cdot x^{-s-1} dx = s \int_0^\infty x^{-s-1} \sum_{n=1}^\infty \frac{\pi(x^{1/n})}{n} dx = s \int_0^\infty J(x) x^{-s-1} dx,$$

as desired. ■

By the Mellin Inversion Theorem, this implies that

$$\boxed{J(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \log \zeta(s) x^s \frac{ds}{s}}$$

for real $a > 1$. Integrating by parts, this becomes

$$\begin{aligned} J(x) &= \frac{1}{2\pi i}\left(\left(\frac{\log\zeta(s)x^s}{s\log x}\right)_{a-i\infty}^{a+i\infty} - \frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\frac{log\zeta(s)}{s}\cdot x^s dx\right) \\ &= -\frac{1}{2\pi i\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\frac{\log\zeta(s)}{s}x^s ds. \end{aligned}$$

The reason why we rewrite Theorem 3.1 using the Mellin inversion theorem is because that neatly brings $\zeta$ inside the integral. Since we can express $\zeta$ in terms of its zeros using Theorems 2.1 and 2.4, we can explicitly solve for $J(x)$, which is what we focus on in the next section.

## 4. Putting everything together

We may take the logarithm of both sides of Definition 2.1 as

$$\log\zeta(s) = \log\xi(s) - \frac{s\log\pi}{2} - \log(s-1) - \log\Gamma(s/2+1).$$

Using Theorem 2.4, this becomes

$$\log\zeta(s) = \sum_\rho \log\left(1-\frac{s}{\rho}\right) + \log\xi(0) - \frac{s\log\pi}{2} - \log(s-1) - \log\Gamma(s/2+1).$$

It is known that $\xi(0) = \frac{1}{2}$, so plugging that in and dividing both sides by $-s$, we get

$$-\frac{\log\zeta(s)}{s} = -\frac{\sum_\rho \log\left(1-\frac{s}{\rho}\right)}{s} + \frac{\log 2}{s} + \frac{\log\pi}{2} + \frac{\log(s-1)}{s} + \frac{\log\Gamma(s/2+1)}{s}.$$

We now plug this into the formula stated at the end of Section 3. For simplicity, we will consider things term by term. The second term gives

$$\frac{1}{2\pi i\log(x)}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\frac{\log(2)}{s}x^s ds = -\frac{\log 2}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{x^s ds}{s} = -\log 2$$

upon integrating by parts in the reverse direction. The third term has derivative 0, so it contributes nothing.

The rest of the terms are not as easy to simplify, so we state each term's contribution without proof. The fourth term gives the "main" term of the explicit formula $\mathrm{Li}(x)$. The fifth term gives the integral term $\int_x^\infty \frac{dt}{t(t^2-1)\log t}$.

Meanwhile, for the first term, it turns out (but we make no effort to prove) that we can swap the summation and integral in the resulting term to get

$$-\sum_\rho \int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\frac{\log\left(1-\frac{s}{\rho}\right)}{s}x^s ds = -\sum_\rho \mathrm{Li}(x^\rho),$$

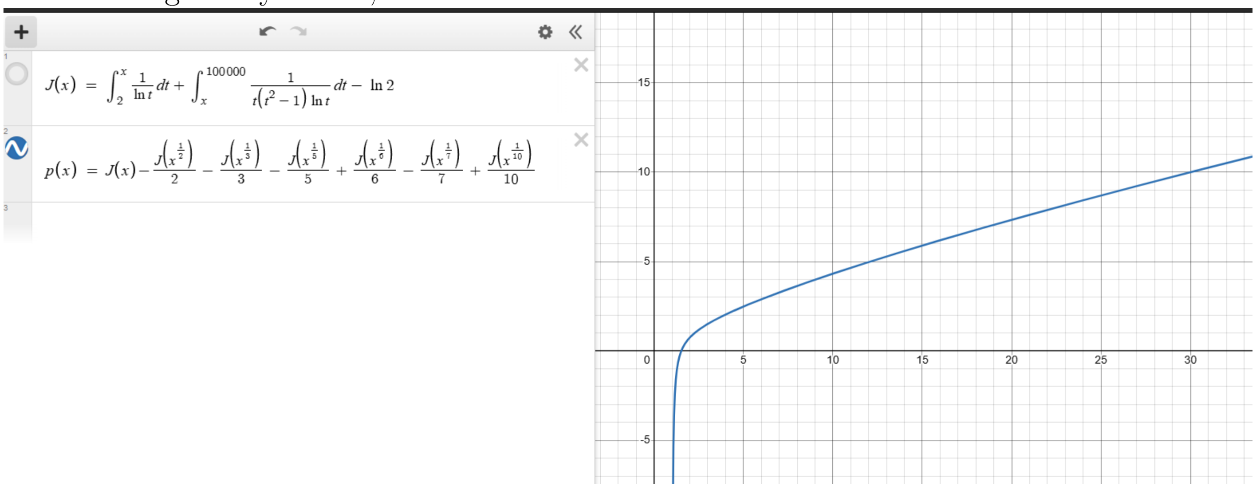
where the last step comes from evaluating the integral in a similar way to the fifth term (which we omit). However, the sum is only conditionally convergent, so we must pair up each zero with its reflection across 1, in particular $1-\rho$. Putting everything together, we arrive at

$$\boxed{J(x) = \mathrm{Li}(x) - \sum_{\rho}(\mathrm{Li}(x^{\rho}) + \mathrm{Li}(x^{1-\rho})) + \int_{x}^{\infty} \frac{dt}{t(t^2-1)\log t} - \log 2},$$

which is exactly what we aimed to prove in this paper.

## 5. Applying the Formula

Now we will put this formula to use in some examples. First, using Theorems 1.3 and 1.4, we graph a rough estimate of $\pi(x)$ in Desmos (in this case called $p(x)$). We use the expansion for $\pi(x)$ up until $\frac{J(x^{1/10})}{10}$ and take the integral in Theorem 1.4 up until 100000, without adding in any zeroes, as shown below:




If we evaluate $p(10000)$, we get that

$$p(10000) \approx 1226.9.$$

A simple calculation yields that the number of primes less than or equal to 10000 is 1229, so $p(x)$ gives a very reasonable estimate for the number of primes less than or equal to $x$. Moreover, we haven't taken the zeroes into account; if we take the upper limit of the integral towards $\infty$, extend the expansion of $\pi(x)$ indefinitely, and take the zeroes into account, as the explicit formula states, we can get an explicit formula for the number of primes.

This brings us to Riemann's hypothesis:

**Conjecture 5.1** (Riemann's Hypothesis)**.** *The nontrivial zeroes of $\zeta$ all have real part $\frac{1}{2}$.*

If we can prove this conjecture, then the terms in Theorem 1.4 involving the zeroes of $\zeta$ will turn out to be particularly nice (because each term just gets paired with its conjugate), so we can end up taming to the function to a great extent, allowing us to accurately approximate the primes

Alas, this conjecture was formed 165 years ago, and still no one has been able to find a proof of it.

## 6. Conclusion

In this paper, we explored Riemann's Explicit Formula for the number of primes less than or equal to $x$. We started the paper by defining the function $J(x)$ and connecting it to the prime-counting function $\pi(x)$. We then stated the explicit formula and began our proof of the theorem. Our first step in the proof was examining the properties of the function $\xi$, which is closely related to $\zeta$. We then related $\zeta$ and $J(x)$ using an integral relation, and finally we plugged in the value of $\zeta(s)$ in terms of $\xi$ to finish the proof.

This remarkable formula is perhaps why the Riemann hypothesis is marked as one of the most important unsolved problems in mathematics. If the Riemann hypothesis is true, then because of the terms involving the zeroes in the explicit formula, we are able to tame $J$, which will allow us to approximate $\pi(x)$ particularly well. This will get us even closer to solving the mystery of the distribution of primes that haunted mathematicians for thousands of years.

## 7. Acknowledgements

The author would like to thank Simon Rubinstein-Salzedo, Matthew Kroesche, and Carson Mitchell for giving insights on this paper and helping correct errors in prior drafts.

## 8. References

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