OTHER ZETA FUNCTIONS

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1. Topic Summary

In this paper, we will discuss different Zeta Functions and their significance regarding analytic number theory, and a couple of other topics. We will introduce first the general requirements for a zeta function, followed by the Riemann and Hurwitz Zeta Functions as the guideline for other zeta functions. Next, we will move on to the significance of the Dedekind Zeta Function in algebraic number theory, and finally the Weil and Dynamical Zeta Functions in ergodic theory.

2. Primary Zeta Functions

First, we need to know what a Zeta function is. Broadly, a zeta function is a function that behaves similarly to the Riemann Zeta function, but can be applied to different areas of mathematics depending on what tools are used to define it.

Definition 2.1. Requirement for a Zeta Function Generally, a Zeta function satisfies the following properties:

1 Meromorphic on all of $\mathbb{C}$
2 Have Dirichlet Series expansions
3 Have Euler Product expansions
4 Satisfy specialized functional equations

Essentially, the combination of conditions 1 through 3 allow us to connect a differentiable function to the distribution of primes, which can be seen through the complex analytic proof of the prime number theorem (which is done primarily by taking the logarithmic derivative of the zeta function and verifying it is holomorphic on $\mathbb{C} \setminus 1$; see 1 for more details), as well as corollaries of the Riemann hypothesis allowing us to predict the distribution of primes exactly.

These four conditions are of course satisfied by the classic example of a zeta function:

Definition 2.2. Riemann Zeta Function The Riemann Zeta function is defined by $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$

As we showed in class, this function has Euler Product given by $\sum_{p \text{ prime}} (1 - p^s)^{-1}$. Furthermore, we can extend the Zeta function to a meromorphic function on all of $\mathbb{C}$; we did

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this on the positive real value half-plane by implementing the Alternating Zeta Function, but this idea can be generalized further.

Finally, the Riemann Zeta Function satisfies many special functional equations; as exercises, we showed the Zeta Reflection Formula:

Theorem 2.3. ***Zeta Reflection Formula*** *For any $s \in \mathbb{R} \setminus \mathbb{N}$, the following identity holds:* $\zeta(s) = \zeta(1-s)(2^s \pi^{s-1} \sin \frac{s\pi}{2} \Gamma(1-s)$

The restriction on s is just to ensure the zeta function and Gamma function do not have poles at our specified values.

In fact, a similar version of this reflection formula can be generalized to $\mathbb{C}$, by replacing the $1-s$ used as inputs on the right hand side with the reflection of s over the line $Re\ s = 1/2$ (which coincides with our previous statement when s is real).

Using Dirichlet series, we also showed that $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$, which is another important algebraic relation of the function. Using different algebraic techniques, generally involving integral transformations we can find other important functional equations, but equations similar to these are less common in comparison to the aforementioned for other zeta functions.

We now introduce our first new zeta function:

Definition 2.4. Hurwitz Zeta Function, see 2 $\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$

Typically, we fix $0 < a \leq 1$, with a typically rational (but this is not a requirement), to have a slightly modified Dirichlet series. For all a satisfying these conditions, $\zeta(s,a)$ satisfies certain conditions for a Zeta function. In particular, it is meromorphic and satifies an analogue of the Zeta Reflection Formula, but only has Euler product decomposition when $a = 1$, as this requires for the leading term of the Dirichlet series to be 1.

This function is useful at evaluating L-functions of Dirichlet characters, as we did in an exercise with $L(1,\chi)$ where χ is the non-trivial character mod 4; many of the techniques used in this exercise can be generalized (by rewriting our L-function in terms of the Hurwitz Zeta Function evaluated at different values of a) to all characters. However, the idea behind the Hurwitz zeta function can be generalized further, which we will look at in the next section.

3. Dedekind Zeta Function

As the Riemann Zeta Function has significant usages and study, mathematicians generalized the function - in the case of the Dedekind Zeta Function, we focus on working with general fields and their rings of integers rather than just that of the positive integers.

Definition 3.1. Dedekind Zeta Function, see 3 Let K be an algebraic number field. Its Dedekind Zeta function is defined (for $s \in \mathbb{C}; Re(s) > 1$) by $\zeta_K(s) = \sum_{I \in O_k} \frac{1}{N_{K/\mathbb{Q}}(I)^s}$, where I ranges through the non-zero ideals of O_k the ring of integers of K, and $N_{K/\mathbb{Q}}$ is the ideal norm of I given by the index of I in O_k. This sum converges when $Re(s) > 1$ regardless of

the number field we observe, so this is a well defined definition. When $K = \mathbb{Q}$, note that all of our definitions align such that the Dedekind Zeta Function equals the Riemann Zeta Function.

We can use this function to prove analogous properties of field extensions of $\mathbb{Q}$, and important facts about prime ideals in these fields. As the generalized version of a Dirichlet series starts with the term 1 (or terms with norm 1), we intuitively should be able to write the zeta function as an Euler product. This is indeed the case:

Euler Product of Dedekind Zeta Function: Let p be a prime ideal, and Re $s > 1$. Then, $\zeta_K(s) = \prod_{p \text{ prime} \subset O_k}(1 - N_{K/\mathbb{Q}}(p)^{-s})^{-1}$

Proving this fact in fact can be done using the techniques used to show the Euler product of the Riemann Zeta Function, combined with from the fact our ring of integers is a Dedekind domain, which is a ring such that any ideal can be represented as the unique product of prime ideals.

A natural application of the Dedekind Zeta function is to quadratic fields, such as $\mathbb{Q}(\sqrt{2})$ or $\mathbb{Q}(i)$. We can in fact evaluate this in terms of the Riemann zeta function and the product of L-functions

Theorem 3.2. *If K is a quadratic extension of $\mathbb{Q}$, then $\frac{\zeta_K(s)}{\zeta(s)} = L(s, \chi)$, where χ is some Dirichlet character (In fact, if K is any abelian extension we can represent this quotient as the product of L-functions)*

Proof. We show this for the quadratic extension case; similar logic can be applied to generalize this proof. First, note that $L(s, \chi) = \prod_{p \text{ prime}}(1 - \chi(p)p^{-s})^{-1}$, so we can reinterpret this problem in terms of Euler products, and thus primes. Every rational prime, when interpreted as elements of $O(K)$ either split, or are fixed; we can select χ such that $\chi(p) = 1$ when p splits, and -1 when it is fixed (0 if the prime ramifies, but we'll ignore this detail for now).

When we rearrange both sides of our initial equation by multiplying by $\zeta(s)$, it is equivalent to show that for each prime $p \in \mathbb{N}$ that is fixed in K that $(1 - p^{-2s})^{-1} = (1 - p^{-s})^{-1}(1 - \chi(p)p^{-s})^{-1}$, and that for each prime that splits we have $(1-p^{-s})^{-1}(1-p^{-s})^{-1} = (1-p^{-s})^{-2}$, which are both naturally true (in the first case, p has norm p^2 by definition of a norm, and similarly in the second case the primes dividing p each have norm p).

When we apply this theorem to $\mathbb{Q}(\sqrt{2})$ our $\chi(x)$ is the character mod 8 that equals 1 when $x = 1, 7 \pmod 8$ and -1 when $x = 3, 5 \pmod 8$, and $\mathbb{Q}(i)$ is just the Jacobi symbol modulo 4 (detecting quadratic residues).

A surprising generalization of our previous theorem, which we will cite but not prove is the following:

Theorem 3.3. *If L is a Galois extension of K, then $\frac{\zeta_L(s)}{\zeta_K(s)}$ is holomorphic, so these zeta functions "divide" one another.*

This theorem is beautiful in a similar way to the Fundamental Theorem of Galois Theory, as it guarantees we can "stack" zeta functions (by their factors in an euler product) akin *to field extensions. For an intuitive introduction to algebraic number theory concepts, I recommend looking at 5.*

4. Weil Zeta Function and Dynamical Zeta Functions

We now look primarily at the Weil Zeta function, and then move onto the significance of dynamical zeta functions, which are defined similarly but will not be as emphasized.

As a review, first:

Definition 4.1. Dynamical System A dynamical system $(M, \mathcal{A}, \mu, T)$, abbreviated as (M, T), is a measure space $(M, \mathcal{A}, \mu)$ equipped with the measure preserving transformation T. If T is ergodic, we refer to (M, T) as an ergodic dynamical system.

Essentially, these are just the spaces we worked with in Ergodic Theory, but formatted slightly differently.

Definition 4.2. Weil Zeta Function, see 6 The Weil Zeta function of a dynamical system (M, f) is defined by $\zeta(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} |\text{Fix } f^m|$, where Fix f^m is the set of fixed points of f^m (which we assume to be finite). Initially, this function was limited to looking at

algebraic varieties on finite fields $\mathbb{F}_{p^n}$ with f being the Frobenius map $x \to x^{p^n}$ (acting on the algebraic closure of this field), but could be extended to consider general dynamical systems. Before looking into this more, we note that the Weil Zeta Function in fact satisfies

the conditions of a zeta function as defined in section 1. In fact, its Euler product form is equal to $\zeta(z) = \prod_P (1 - z^P)^{-1}$, where our product is over all periodic orbits P and $|P|$ is the period of P; showing this follows from evaluating the logarithm of the zeta function and using Taylor series expansion, which is valid whenever $|z| < 1$.

This function generally used as a way to count periodic orbits of a map with weighting determined by z. In particular, it sees usage by letting f be a diffeomorphism (differentiable + inverse differentiable homeomorphism) on M, as it helps us to identify our fixed/periodic points. Additionally, when f is hyperbolic, its corresponding Zeta Function is surprisingly rational (Ruelle), making it especially useful when dealing with hyperbolic space. However, these require more background in differential geometry than we currently have, so we will not look into it further.

The definition of a Dynamical Zeta function is essentially a generalized version of the Weil Zeta Function:

Definition 4.3. Dynamical Zeta Functions Let (M, f) be a dynamical system, $g : M \to \mathbb{C}$ be a weight function and consider the power series: $\zeta(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \text{Fix } f^m} \prod_{k=0}^{m-1} g(f^k x)$. When $g = 1$ everywhere, this definition coincides with that of the Weil Zeta Function, and

similarly satisfies the necessary properties of a Zeta function. These functions thus serve a similar purpose to that of the Weil Zeta Function, but use a weight function to generalize to situations where what we want to count is not solely the number of fixed points of a transformation.

5. Bibliography

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