AN OVERVIEW OF THE WIENER-IKEHARA THEOREM

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Abstract. In this paper we will look at a proof of the Wiener-Ikehara Theorem using both tricks in Analytic Number Theory and Complex Analysis. We will also discuss some applications of the Wiener-Ikehara Theorem and its relations with the Prime Number Theorem and other applications. We will closely follow A. Vatwani in this paper.

1. Introduction

The Wiener-Ikehara theorem is one of many Tauberian Theorem. Let's consider Abel's Theorem. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series centered at $x = 0$ and with radius of convergence 1. Abel's theorem states that if the boundary point ($|x| = 1$) then it is continuous at that point. Essentially, a Tauberian Theorem are converses of Abel sums. They derive their names from a theorem of Alfred Tauber, which states that if

$$\lim_{n \to 1^-} \sum_{n=0}^{\infty} a_n x^n = A$$

and the series grows at $a_n = o(1/n)$ are satisfied, then

$$\sum_{n=0}^{\infty} a_n = A$$

holds. In this paper we will explore the Wiener-Ikehara Theorem which is used to estimate the sum of the partial sums of a Dirichlet Series. The theorem was introduced by Ikehara in 1931, and it generalizes a result from Landau who applied a Tauberian result obtained by Wiener. The Wiener-Ikehara can directly be used to prove the Prime Number Theorem and in 1980, Newman came up with an ingenious proof. We will draw on his results in this paper to prove the Wiener-Ikehara Theorem.

2. Newman's Theorem

Newman came up with an analytic theorem that is essential in proving the Wiener-Ikehara Theorem. To give an broad overview, Newman introduced a novel kernel into the integral in the Cauchy-Residue Theorem which makes proving Tauberian Theorems easier as they are usually quite involved. Before we state the theorem and the proof, let's define some terminology.

Definition 2.1. A function is said to be **entire** if it is analytic on all finite points of $\mathbb{C}$.

Definition 2.2. A countour is said to be **positive** if it travels counter-clockwise and said to be **negative** if it travels clockwise.

Definition 2.3. Cauchy's Residue Theorem states that

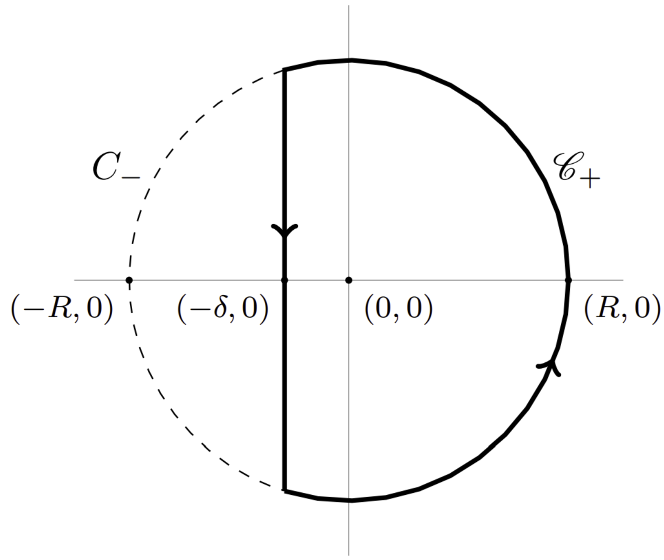
$$\int_C f(z)dz = 2\pi i \cdot (\text{Sum of Residues inside C})$$

where a residue is defined to be points with singularity.

Now let's look at Newman's analytic theorem and it's proof.

Theorem 2.4. *For $t \geq 0$, let $f(t)$ be a bounded and locally integrable function and let $g(s) := \int_0^\infty f(t)e^{-st}dt$ for $Re(s) > 0$. If $g(s)$ has an analytic continuation to $Re(s) \geq 0$, then $\int_0^\infty f(t)dt$ exists and equals $g(0)$.*

Proof. We will denote $g_T(s) = \int_0^T f(t)e^{-st}dt$ for all $T > 0$. Clearly $g_T(s)$ is an entire function. Rewording the statement in theorem, we need to show $\lim_{T\to\infty} g_T(0) = g(0)$. Denote $Re(s) = \sigma$ and consider the positively oriented contour, $\mathscr{C}$, shown below.



We choose $\delta > 0$ small enough so that $g(s)$ is analytic on $\mathscr{C}$. Also define $\mathscr{C}_+ = \mathscr{C} \cap \{s : \sigma > 0\}$ and $\mathscr{C}_- = \mathscr{C} \cap \{s : \sigma < 0\}$. We also will denote C_- as the semicircle where $\sigma < 0$. Introducing Newman's novel kernel we know that

$$I_{\mathscr{C}} := \frac{1}{2\pi i} \int_{\mathscr{C}} \left(1 + \frac{s^2}{R^2}\right) e^{sT}(g(s) - g_T(s))\frac{1}{s}ds.$$

Note that 0 is a pole so we can calculate the residue at 0:

$$\lim_{s\to 0}(s - 0)(g(s) - g_T(s)) \cdot e^{sT}\left(1 + \frac{s^2}{R^2}\right) \cdot \frac{1}{s} = g(0) - g_T(0).$$

By Cauchy's Theorem we get $I_{\mathscr{C}} = g(0) - g_T(0)$. We will find the contributions from $I_{\mathscr{C}_+}$ and $I_{\mathscr{C}_-}$ to $I_{\mathscr{C}}$ using big O notation where R, T, σ are treated as variables.

Let $M = \sup_{t\geq 0} |f(t)|$. On $\mathscr{C}_+$ as $\sigma > 0$, we have

$$|g(s) - g_T(s)| = \left|\int_T^\infty f(t)e^{-st}dt\right| \leq M \int_T^\infty e^{-st}dt << \frac{e^{-\sigma t}}{\sigma}.$$

We can cleverly let $s = Re^{i\theta}$ and $\sigma = R\cos(\theta)$ on $\mathscr{C}_+$ and use this to estimate Newman's Kernel. Doing so we get

$$\left|\left(1 + \frac{s^2}{R^2}\right) e^{sT} \cdot \frac{1}{s}\right| = e^{\sigma T}\left|\frac{1}{Re^{i\theta}} + \frac{e^{i\theta}}{R}\right| = e^{\sigma T}\left|\frac{2\cos(\theta)}{R}\right| << e^{\sigma T}\frac{|\sigma|}{R^2}.$$

We will refer to this estimate as the Kernel Estimate. Since the length of $\mathscr{C}_+$ is πR the contribution to $I_{\mathscr{C}}$ is

$$|I_{\mathscr{C}}| << \frac{1}{R^2}\left|\int_{\mathscr{C}_+} ds\right| << \frac{1}{R}.$$

Hence this is just $O\left(\frac{1}{R}\right)$ using big-O notation.

Dealing with $\mathscr{C}_-$ is a little more tricky so we will consider $g(s)$ and $g_T(s)$ separately. Define

$$I_1 := \frac{1}{2\pi i}\int_{\mathscr{C}_-} g_T(s)e^{sT}\left(1+\frac{s^2}{R^2}\right)\cdot\frac{1}{s}ds.$$

Note that integrating over C_- is the same as integrating over $\mathscr{C}_-$ because $g_T(s)$ is entire and the integrand is analytic at $\sigma < 0$. So

$$I_1 = \frac{1}{2\pi i}\int_{C_-} g_T(s)e^{sT}\left(1+\frac{s^2}{R^2}\right)\cdot\frac{1}{s}ds.$$

Like before but with $\sigma < 0$ we get

$$|g_T(s)| = \left|\int_0^T f(t)e^{-st}dt\right| \leq M\int_0^T e^{-\sigma t}dt << \frac{e^{-\sigma T}}{|\sigma|}.$$

Once again we can let $s = Re^{i\theta}$ and $\theta = R\cos(\theta)$ but this time on C_- and use this to estimate Newman's Kernel. Since the length of the path of C_- is the same as the length of the path of $I_{\mathscr{C}_+}$ the contribution from $|I_1|$ to $|I_{\mathscr{C}}|$ is the same as the contribution from $|I_{\mathscr{C}_+}|$ to $|I_{\mathscr{C}}|$. Thus $|I_1| << \frac{1}{R}$. Note that this is just $O\left(\frac{1}{R}\right)$ using big-O notation.

Now let's look at $g(s)$. Define

$$I_2 := \frac{1}{2\pi i}\int_{\mathscr{C}_-} g(s)e^{sT}\left(1+\frac{s^2}{R^2}\right)\cdot\frac{1}{s}ds.$$

We will split $\mathscr{C}_-$ by the arcs and the line segment. Let's first find the contribution to $|I_{\mathscr{C}}|$ that the arcs contribute. Since $\mathscr{C}_-$ is contained in a compact set on which $g(s)$ is analytic on, $|g(s)|$ can be bounded by R. By using our Kernel Estimate again, we can see that the integrand of I_2 is of order $|\sigma|e^{\sigma T}$ as $T\to\infty$ with an implicit constant depending on R. Since $\sigma < 0$ is this region and letting $\sigma T = x$,

$$|\sigma|e^{\sigma T} = \frac{|x|}{T}e^{-x} \leq \frac{e^{-1}}{T}$$

the last inequality coming from the fact that the maximum of $|x|e^{-x}$ is e^{-1}. So the integrand can be bounded by $O_R\left(\frac{1}{T}\right)$. So the contribution of the arcs from $\mathscr{C}_-$ is $O_R\left(\frac{1}{T}\right)$ as $T\to\infty$ with the arclength absorbed by the implicit constant depending on R. For the line segment we know $\sigma = -\delta$ and hence the contribution from the segment to $|I_{\mathscr{C}}|$ is just $O_R(e^{-\delta T})$. Hence the contribution to $|I_{\mathscr{C}}|$ from $|I_2|$ is just $O_R\left(\frac{1}{T}\right) + O_R(e^{-\delta T})$.

Now we can put everything together. Hence

$$\begin{aligned}|g(0) - g_T(0)| = |I_{\mathscr{C}}| &= |I_{\mathscr{C}_+}| + |I_1| + |I_2| \\ &<< O\left(\frac{1}{R}\right) + O_R\left(\frac{1}{T}\right) + O_R(e^{-\delta T}).\end{aligned}$$

Since R can be made as small as possible, this completes the proof. □

3. Tauberian Theorem

Now let's start proving the Wiener-Ikehara Theorem.

Theorem 3.1. *Let $G(s) = \sum_{n=1}^{\infty}\frac{b_n}{n^s}$ be a Dirichlet series with non-negative coefficients, satisfying*

- *$G(s)$ is absolutely convergent for $Re(s) > 1$*

- *$G(s)$ extends meromorphically to the half-plane $Re(s) > 1$ having no poles except possible $s = 1$ with $Res(s) = R$.*
- *$B(x) = \sum_{n\leq x} b_n = O(x)$.*

Then as $x \to \infty$ we get $B(x) = Rx + o(x)$.

A naturally starting point to prove any Tauberian Theorem is to try to use Abel's trick. After all Tauberian Theorems are converses of Abel's Theorem. Abel's trick states that, for $Re(s) > 1$,

$$G(s) = s\int_1^\infty \frac{B(x)}{x^{s+1}}dx.$$

As a side note, generally, we don't need the last condition for the Wiener-Ikehara Theorem but we will keep it as it simplifies the problem significantly. Our goal is to show that $B(x) \sim x$ as $x \to \infty$.

Proof. Without loss of generality let $R > 0$. Note that for $Re(s) > 1$, we have

$$\frac{G(s)}{s} - \frac{1}{s-1} = \int_1^\infty \frac{B(x)}{x^{s+1}} - \frac{1}{x^s}dx.$$

This can be verified with basic calculus and algebra. We can do some re-indexing where we send s to $s+1$ and do some change of variables where $x = e^u$. Since $dx = e^u du$, doing our change of variables we get

$$\frac{G(s+1)}{s+1} - \frac{1}{s} = \int_0^\infty \frac{B(e^u) - e^u}{e^{u(s+2)}}e^u du = \int_0^\infty \frac{B(e^u) - e^u}{e^u} \cdot e^{-su}du.$$

The reason we do these changes of variables is to use Newman's Theorem so with that motivation, let's define

$$f(s) := \frac{B(e^u) - e^u}{e^u}.$$

We know that $f(s)$ is bounded by property (c) in the statement of the theorem and it also has an analytic continuation due to property (b) in the statement of the theorem. Once again by change of variables, namely $e^u = t$ and Newman's Theorem, we know that

$$\int_0^\infty \frac{B(e^u) - e^u}{e^u} = \int_1^\infty \frac{B(t) - t}{t^2}dt$$

converges. So we will show that $B(x) \sim x$ as $x \to \infty$. We will do a proof my contradiction. So we will assume that $\lim_{x\to\infty} \frac{B(x)}{x} \neq 1$ if the limit exists or it does not even exist. This means that

$$\limsup_{x\to\infty} \frac{B(x)}{x} > 1 \text{ or } \liminf_{x\to\infty} \frac{B(x)}{x} < 1.$$

Let's handle the first case. The other case follows the exact same argument as shown below. By definition of supremum there exists some $\lambda > 1$ such that $B(x) \geq \lambda x$. Since x has no bounds and $B(x)$ an increasing function, we have

$$\int_x^{\lambda x} \frac{B(t) - t}{t^2}dt \geq \int_x^{\lambda x} \frac{\lambda x - t}{t^2}dt.$$

Once again we use change of variables where $t = vx$. Doing so we get

$$\int_1^\lambda \frac{\lambda x - vx}{(vx)^2}xdv = \int_1^\lambda \frac{\lambda - v}{v^2}dv = q(\lambda)$$

where $q(\lambda)$ is just a quantity depending on λ. This means that

$$\left|\int_x^{\lambda x} \frac{B(t) - t}{t^2}\right| = \left|\int_x^\infty \frac{B(t) - t}{t^2} - \int_{\lambda x}^\infty \frac{B(t) - t}{t^2}\right| = q(\lambda).$$

Note that $\int_x^\infty \frac{B(t)-t}{t^2}$ and $\int_{\lambda x}^\infty \frac{B(t)-t}{t^2}$ are just tails of the convergent integral $\int_1^\infty \frac{B(t)-t}{t^2}$. So for a fixed λ, we can make the tails of the convergent integral as small as we want, and hence we reach a contradiction. □

This completes the proof of the Wiener-Ikehara Theorem. Clearly this was much easier to proof with the help of Newman's Theorem. We can also address the case where the coefficients of our Dirichlet Series are complex numbers.

Theorem 3.2. *Let* $F(s) = \sum_{n=1}^\infty \frac{a_n}{n^s}$ *be a Dirichlet series with complex coefficients. Let* $A(x) = \sum_{n\leq x} a_n$. *Let* $G(s) = \sum_{n=1}^\infty \frac{b_n}{n^s}$ *be a Dirichlet series with non-negative coefficients, such that*

- $|a_n| \leq b_n$ *for all* n
- $G(s)$ *is absolutely convergent for* $Re(s) > 1$
- $G(s)$ *extends meromorphically to the half-plane* $Re(s) > 1$ *having no poles except possible* $s = 1$ *with* $Res(s) = R$.
- $B(x) = \sum_{n\leq x} b_n = O(x)$.

Then as $x \to \infty$ *we get* $A(x) = rx + o(x)$.

Proof. If the values of a_n are real then we can just consider the Dirichlet Series $G(s) - F(s)$ which has non-negative coefficients and gives

$$\sum_{n\leq x}(b_n - a_n) = (R - r)x + o(x)$$

when $x \to \infty$ as desired. If the coefficients a_n are not real, then we can define a new function

$$I(x) := \sum_{n=1}^\infty \frac{\overline{a_n}}{n^s}.$$

We can cleverly write

$$F = \frac{F + I}{2} + i\left(\frac{F - I}{2i}\right)$$

and then use the results from Theorem 3.1 on the real part and the imaginary part. This satisfies the conditions of both Theorem 3.1 and Theorem 3.2 and hence completes this proof. □

4. Applications on the Prime Number Theorem

Now that we have taken a look at the proof of the Wiener-Ikehara Theorem, let's look at some applications. The Wiener-Ikehara Theorem has direct applications to the Prime Number Theorem.

The Prime Number Theorem states that the number of primes less than or equal to x, denoted by $\pi(x)$, satisfies

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as} \quad x \to \infty.$$

To see how the Wiener–Ikehara theorem applies, consider the Chebyshev function $\psi(x)$ defined by

$$\psi(x) = \sum_{p^k\leq x} \log p,$$

where the sum is over all prime powers $p^k \leq x$. The Laplace transform of $\psi(x)$, denoted by $\mathcal{L}\{\psi(x)\}(s)$, is closely related to the logarithm of the Riemann zeta function $\zeta(s)$.

From properties of $\zeta(s)$ and its relation to prime numbers, it can be shown that

$$\mathcal{L}\{\psi(x)\}(s) = -\frac{\zeta'(s)}{\zeta(s)}.$$

The Riemann Hypothesis implies that $\zeta(s)$ has no zeros on the line $\Re(s) = 1$, but assuming only that $\zeta(s)$ has a simple pole at $s = 1$ with residue 1, and no other poles or zeros on $\Re(s) \geq 1$, we get:

$$-\frac{\zeta'(s)}{\zeta(s)} \sim \frac{1}{s-1} \quad \text{as} \quad s \to 1.$$

By applying the Wiener–Ikehara theorem, it follows that

$$\psi(x) \sim x \quad \text{as} \quad x \to \infty.$$

From this result and properties of $\psi(x)$, the Prime Number Theorem $\pi(x) \sim \frac{x}{\log x}$ is proved.

5. Applications in Probabilistic Number Theory

5.1. **Distribution of Prime Gaps.** The gaps between consecutive primes, denoted by $g_n = p_{n+1} - p_n$, where p_n is the n-th prime, have been extensively studied. Probabilistic models of these gaps often rely on average-case results.

The Wiener–Ikehara theorem can be used to find the distribution of prime gaps by considering the sum of prime gaps up to a certain number x. Let $G(x) = \sum_{p_n \leq x} g_n$. The properties of the Laplace transform of $G(x)$ can be used to derive asymptotic results about $G(x)$, which tell us about the average prime gap.

5.2. **Average Order of the Euler Totient Function.** The Euler totient function $\varphi(n)$, which counts the number of integers up to n that are coprime to n, is another important function in number theory. The average order of $\varphi(n)$ can be studied using the Wiener–Ikehara theorem.

Let $F(x) = \sum_{n \leq x} \varphi(n)$. The Dirichlet series for $\varphi(n)$ is given by

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)},$$

where $\zeta(s)$ is the Riemann zeta function. By analyzing the poles and residues of this series, and applying the Wiener–Ikehara theorem gives us

$$F(x) \sim \frac{3}{\pi^2} x^2.$$

This result gives the average value of $\varphi(n)$, providing insights into its distribution.

5.3. **Random Multiplicative Functions.** Random multiplicative functions are functions $f(n)$ such that $f(mn) = f(m)f(n)$ for relatively prime m and n, and where $f(p)$ (for primes p) are independent random variables. The mean value and distribution of such functions can be found as follows.

Consider a random multiplicative function $f(n)$ with $\mathbb{E}[f(p)] = 0$ and $\mathbb{E}[|f(p)|^2] = 1$. The sum $F(x) = \sum_{n \leq x} f(n)$ can be analyzed using the Wiener–Ikehara theorem. The Laplace transform techniques can be used to find the expected value and variance of $F(x)$:

$$\mathbb{E}[F(x)] = 0 \quad \text{and} \quad \mathrm{Var}(F(x)) \sim x \log\log x.$$

This allows for the discovery of the typical behavior of random multiplicative functions.

5.4. **Moments of Number-Theoretic Functions.** The Wiener–Ikehara theorem also assists in studying the moments of number-theoretic functions, such as the sum of k-th powers of divisor functions. Let $d_k(n)$ denote the number of ways n can be written as a product of k factors. The average order of $d_k(n)$ can be obtained by examining its associated Dirichlet series and applying the Wiener–Ikehara theorem to gain insight on asymptotic results.

For instance, the sum $D_k(x) = \sum_{n\leq x} d_k(n)$ has the asymptotic form

$$D_k(x) \sim C_k x(\log x)^{k-1},$$

where C_k is a constant depending on k. Such results allow one to find the distribution and average properties of $d_k(n)$.

6. Applications to Arithmetic Functions

6.1. **The Divisor Function.** The divisor function $d(n)$ counts the number of divisors of n. To study its average order, consider the sum $D(x) = \sum_{n\leq x} d(n)$. The Dirichlet series associated with $d(n)$ is given by $\zeta(s)^2$, where $\zeta(s)$ is the Riemann zeta function.

- The Dirichlet series $\sum_{n=1}^{\infty} \frac{d(n)}{n^s}$ can be written as:

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^s} = \zeta(s)^2,$$

  which converges for $\Re(s) > 1$.
- The series $\zeta(s)^2$ has a pole of order 2 at $s = 1$. We know that:

$$\zeta(s) = \frac{1}{s-1} + \gamma + \mathcal{O}(s-1),$$

  where γ is the Euler-Mascheroni constant. Hence,

$$\zeta(s)^2 = \left(\frac{1}{s-1} + \gamma + \mathcal{O}(s-1)\right)^2 = \frac{1}{(s-1)^2} + \frac{2\gamma}{s-1} + \mathcal{O}(1).$$

- The residue of the double pole at $s = 1$ is relevant for the asymptotic behavior. Specifically, the leading term is:

$$\zeta(s)^2 \sim \frac{1}{(s-1)^2}.$$

- Applying the Wiener–Ikehara theorem to $D(x)$, we use the fact that the Laplace transform $\mathcal{L}\{D(x)\}(s)$ has a pole of order 2 at $s = 1$. Thus,

$$D(x) \sim x\log x + (2\gamma - 1)x \quad \text{as} \quad x \to \infty.$$

This shows the average order of the divisor function and how how the number of divisors of integers grows on average.

6.2. **The Sum of Divisors Function.** The sum of divisors function $\sigma(n)$ is defined as the sum of all positive divisors of n. The Dirichlet series for $\sigma(n)$ is given by $\zeta(s)\zeta(s-1)$.

- The Dirichlet series $\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s}$ can be written as:

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \zeta(s)\zeta(s-1),$$

  which converges for $\Re(s) > 1$.

- The series $\zeta(s)\zeta(s-1)$ has poles at $s = 1$ and $s = 2$. Near $s = 1$, we have:
$$\zeta(s-1) = \frac{1}{s-2} + \gamma + \mathcal{O}(s-2),$$
and thus,
$$\zeta(s)\zeta(s-1) \sim \frac{1}{(s-1)(s-2)}.$$
- The residue at $s = 1$ is crucial for determining the asymptotic behavior of $S(x) = \sum_{n \leq x} \sigma(n)$. Near $s = 2$,
$$\zeta(s-1)\zeta(s) \sim \frac{1}{s-1} \cdot \frac{1}{s-1} = \frac{1}{(s-1)^2}.$$
- Applying the Wiener–Ikehara theorem, we conclude that:
$$S(x) \sim \frac{\pi^2}{6} x^2 \quad \text{as} \quad x \to \infty.$$

This result gives the average value of the sum of divisors function, showing that $\sigma(n)$ grows quadratically on average.

6.3. **The Euler Totient Function.** The Euler totient function $\varphi(n)$ counts the number of integers up to n that are coprime to n. The Dirichlet series associated with $\varphi(n)$ is $\frac{\zeta(s-1)}{\zeta(s)}$.

- To study the average order of $\varphi(n)$, consider the Dirichlet series:
$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}.$$
- The function $\zeta(s-1)$ has a simple pole at $s = 2$, and $\zeta(s)$ has a simple pole at $s = 1$. Hence,
$$\frac{\zeta(s-1)}{\zeta(s)} \sim \frac{1}{s-2} \cdot (s-1) = \frac{1}{s-2}.$$
- Analyzing the pole at $s = 2$, we apply the Wiener–Ikehara theorem to the sum $F(x) = \sum_{n \leq x} \varphi(n)$, leading to:
$$F(x) \sim \frac{3}{\pi^2} x^2 \quad \text{as} \quad x \to \infty.$$

This result shows that $\varphi(n)$ also grows quadratically on average.

7. Other Arithmetic Functions

The Wiener–Ikehara theorem also finds applications in studying other arithmetic functions, such as the sum of k-th powers of divisor functions. Let $d_k(n)$ denote the number of ways n can be written as a product of k factors. The average order of $d_k(n)$ can be obtained by analyzing its Dirichlet series.

- The sum $D_k(x) = \sum_{n \leq x} d_k(n)$ has the Dirichlet series:
$$\sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} = \zeta(s)^k,$$
which converges for $\Re(s) > 1$.
- The series $\zeta(s)^k$ has a pole at $s = 1$ of order k, and the residue depends on k. For large x,
$$\zeta(s)^k \sim \frac{1}{(s-1)^k}.$$

- Applying the Wiener–Ikehara theorem, we get:

$$D_k(x) \sim C_k x(\log x)^{k-1},$$

where C_k is a constant depending on k.

8. Acknowledgements

We would like to thank Dr. Simon Rubestine Salezdo for providing me with the opportunity to write this paper and TA Eric Hallman for reading through this and providing feedback to improve the work.

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