RIEMANN'S EXPLICIT FORMULA

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Abstract. In this paper we give a sketch of the proof of the Riemann-von Mangoldt explicit formula and discuss its consequences on the distribution of the prime numbers.

1. Introduction

In Riemann's 1859 paper "On the Number of Primes Less Than a Given Magnitude" [5], he sketched a proof for an explicit formula for the prime counting function $\pi(x)$ in terms of the zeros of the Riemann zeta function $\zeta(s)$. This formula was later proven rigorously by von Mangolt ([4]), who reformulated it in terms of the Chebyshev function

$$\psi(x) = \sum_{p^m \leq x} \log p.$$

In this form, the explicit formula states

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log(1 - x^{-2}),$$

where ρ ranges over the non-trivial zeros of ζ. This is one of the most important formulas in analytic number theory, giving a precise relationship between the distribution of the primes and the zeros of the zeta function. It has many consequences, notably it gives incredibly precises estimates on $\pi(x)$, and implies that the prime number theorem is equivalent to the non-vanishing of $\zeta(s)$ on the line $\Re(s) = 1$.

In this paper we will give an outline of the proof of this formula, leaving out the delicate complex analysis arguments necessary for a rigorous proof.

2. Preliminaries in Complex Analysis

In this section we give a list of definitions and theorems in complex analysis we will need for studying the zeta function. For more details and proofs of the theorems, see any textbook on complex analysis, for example [6].

Definition 1. Let f be a function defined on a neighborhood of a point $z \in \mathbb{C}$. The *derivative* of f at z is defined as the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h},$$

and is denoted as $f'(z)$, provided this limit exists,.

Date: May 30, 2024.

While the above definition looks very similar to the definition of the derivative of a function of a real variable, it is notable stronger; the limit above is two-dimensional, so the quotient $\frac{f(z+h)-f(z)}{h}$ must approach the same value as $h \to 0$ along any path in the complex plane. In particular, by considering the limit as $h \to 0$ along the real axis and along the imaginary axis, we obtain the following:

Theorem 2. *(Cauchy Riemann Equations) Let $f(x+iy) = u(x,y)+iv(x,y)$ where $u, v : \mathbb{R}^2 \to \mathbb{R}$. If f is differentiable at $z_0 = x_0 + iy_0$ then*

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \quad \text{and} \quad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0).$$

Additionally,

$$f'(z) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0).$$

If the partial derivatives of u and v are continuous then the converse also holds.

Definition 3. A complex function f is said to be *analytic* on an open set $A \subset \mathbb{C}$ if f is differentiable at each point $z_0 \in A$. f is said to be analytic at a point z_0 if f is analytic in some open set containing z_0.

Theorem 4. *(Uniqueness of Analytic Continuation) Let D_1 and D_2 be connected open sets with nonempty intersection, and let f_1 be analytic on D_1. If f_2 is an analytic function on D_2 such that $f_1 = f_2$ on $D_1 \cap D_2$, then f_2 is called an analytic continuation of f_1 to D_2. Further, if such an analytic continuation of f_1 to D_2 exists then it is unique.*

Definition 5. A function f is said to have *a zero of order m* at z_0 if and only if, for all z in some neighborhood of z_0,

$$f(z) = (z - z_0)^m g(z),$$

where g is analytic at z_0 and $g(z_0) \neq 0$.

Definition 6. A function f has a *pole of order m* at z_0 if and only if, for all $z \neq z_0$ in some neighborhood of z_0,

$$f(z) = \frac{g(z)}{(z - z_0)^m},$$

where g is analytic at z_0 and $g(z_0) \neq 0$.

A pole of order 1 is also called a *simple pole*.

Definition 7. A function f is said to be *meromorphic* on an open set D if and only if f is analytic on all of D except for possibly a set of isolated points, which are the poles of f.

An important special case of analytic continuation is when a function f has an analytic continuation to a function $\widetilde{f}$ meromorphic on $\mathbb{C}$. In this case it follows from Theorem 4 that this analytic continuation of f is maximal in the sense that for any other analytic continuation g of f, for all z such that $g(z)$ is defined, $\widetilde{f}$ is also defined and $g(z) = \widetilde{f}$.

Example 8. The *gamma function* is defined by the integral

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt,$$

for all $\Re(s) > 0$. Further, it is analytic on this domain. It is easily verified via integration by parts that Γ satisfies the functional equation

$$\Gamma(s+1) = s\Gamma(s)$$

for all $\Re(s) > 0$. We can use this to construct an analytic continuation of Γ. For all $\Re(s) > -m$, and $s \neq 0, -1, -2, \dots$, we define

$$\Gamma(s) = \frac{\Gamma(s+m)}{s(s+1)\cdots(s+m-1)}.$$

It is easy to see that this uniquely defines a function Γ everywhere except nonpositive integers, and that this extended function still satisfies the functional equation $\Gamma(s+1) = s\Gamma(s)$. Further, this extension is analytic because given any point $s_0 \neq 0, -1, -2, \dots$ we can choose some integer $m < -\Re(s_0)$ and write

$$\Gamma(s) = \frac{\Gamma(s+m)}{s(s+1)\cdots(s+m-1)}.$$

This is analytic at s_0 because the functions $\frac{1}{s}, \dots, \frac{1}{s+m-1}$, and $\Gamma(s+m)$ are analytic at s_0. Further, Γ has a simple pole at each nonpositive integer $-m$ because in some neighborhood of $-m$, we can write

$$\Gamma(s) = \left(\frac{\Gamma(s+m+1)}{s(s+1)\cdots(s+m-1)}\right)\frac{1}{s+m} = \frac{g(s)}{s+m},$$

for all $s \neq -m$, where g is analytic and nonzero at $-m$. Thus, we have found an analytic continuation of Γ to a meromorphic function on $\mathbb{C}$ with simple poles at the nonpositive integers. It can also be shown that Γ has no zeros on all of $\mathbb{C}$ (see [1]).

Definition 9. A *smooth curve* is a function $z : [a,b] \to \mathbb{C}$ which has a nonzero derivative everywhere and is injective except for possibly $z(a) = z(b)$.

The imagine of z, denoted γ, is also sometimes called a smooth curve. Since z is injective and differentiable in this definition, it induces one of two possible orderings on the points of γ. A smooth curve γ along with a specified ordering is called a *directed smooth curve.*

Definition 10. Let γ be a directed smooth curve and let z be a parametrization of γ with its specified order. The *contour integral* of f over γ is defined to be

$$\int_\gamma f(z)\, dz = \int_a^b f(z(t))z'(t)\, dt.$$

Using the change of variables formula, it can be shown this integral is the same for all possible choices of z. Thus, this integral only depends on the directed smooth curve γ. If $-\gamma$ denoted the same set of points as γ but with the opposite order, then

$$\int_{-\gamma} f(z)\, dz = -\int_\gamma f(z)\, dz$$

for any f.

A curve which is made up of piecing together finitely many directed smooth curves $\gamma_1, \dots, \gamma_n$ in a way which agrees with their ordering is called a contour, denoted Γ. It particular, this requires that the last point of γ_i is the same as the first point of γ_{i+1}. If this is the case, we write $\Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$. A contour integral over Γ is defined as

$$\int_\Gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \cdots + \int_{\gamma_n} f(z)\,dz.$$

Theorem 11. *Suppose f is continuous on a connected open set D and has an antiderivative F on all of D. Then, for any contour Γ lying in D with first point a and last point b, we have*

$$\int_\Gamma f(z)\,dz = F(b) - F(a).$$

A contour Γ whose endpoints match is called a *closed contour*. It is called a *simple closed contour* if it has no other multiple points. If Γ is a simple closed contour, it separated the complex plane into an interior and exterior region by Jordan's Theorem. A simple closed contour Γ is said to be *positively oriented* if its interior is always on the left along the parameterization of Γ.

Theorem 12 (Laurent Series)**.** *Let f be analytic in the annulus $r > |z - z_0| < R$. Then f can be expressed as the sum of two series*

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} a_{-j} (z - z_0)^{-j},$$

both series converging in the annulus, and converging uniformly in any closed subannulus $r > \rho_1 \leq |z - z_0| \leq \rho_2 < R$. The coefficients a_j are given by

$$a_j = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^{j+1}}\,d\zeta,$$

where Γ is any positively oriented simple closed contour lying in the annulus and containing z_0 in its interior.

The coefficient a_{-1} is espeically important.

Definition 13. If f has a pole at z_0, then the coefficient a_{-1} in the Laurent expansion of f at z_0 is called the *residue* of f at z_0 and is denoted by

$$\operatorname{Res}(f, z_0) \qquad \text{or} \qquad \operatorname{Res}(z_0).$$

The following very important theorem relates contour integrals of a function f to the singularities of f in the interior of Γ.

Theorem 14 (Cauchy's Residue Theorem)**.** *If Γ is a simple closed positively oriented contour and f is analytic inside and on Γ except at points $z_1, z_2, \ldots, z_n$ inside Γ, then*

$$\int_\Gamma f(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(z_j).$$

3. Definition of the Riemann Zeta Function

The Riemann zeta function, denoted $\zeta(s)$, is a function of a complex variable s, defined by the sum

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots ,$$

which converges for $\Re(s) > 1$. However, this is not the important part of the zeta function for studying prime numbers. Indeed, the distribution of the prime numbers is related to the zeros of ζ, but there are no zeros with $\Re(s) > 1$. We thus need to find an analytic continuation for ζ. We start by writing

$$\begin{aligned}\zeta(s) - \frac{2}{2^s}\zeta(s) &= \left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots\right) - \left(\frac{2}{2^s} + \frac{2}{4^s} + \frac{2}{6^s} + \cdots\right) \\ &= \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots \\ &= \zeta_a(s),\end{aligned}$$

where

$$\zeta_a(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots$$

is the alternating zeta function. Thus, we have

$$\zeta(s) = \left(1 - \frac{1}{2^{s-1}}\right)^{-1} \zeta_a(s).$$

The point of doing this is that $\zeta_a(s)$ converges for all $\Re(s) > 0$ so we can use the above equation to extend $\zeta(s)$ to the half-plane $\Re(s) > 0$ except $s = 1$. Both $\zeta_a(s)$ and $\left(1 - \frac{1}{2^{s-1}}\right)^{-1}$ are analytic on this domain, so $\zeta(s)$ is too. Additionally, ζ_a is analytic and nonzero at 1, and the function $\left(1 - \frac{1}{2^{s-1}}\right)^{-1}$ has a simple pole at $s = 1$, so it follows that ζ has a simple pole at $s = 1$. In fact, we can compute that

$$\lim_{s \to 1} (s-1)\zeta(s) = \frac{s-1}{1 - \frac{1}{2^{s-1}}}\zeta_a(s) = \frac{1}{\ln 2}\zeta_a(1) = 1.$$

The zeta function can actually be extended even more. It can be shown (see [2]) that for all $0 < \Re(s) < 1$, $\zeta(s)$ satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{s\pi}{2} \Gamma(1-s)\zeta(1-s) \tag{15}$$

Since we have already defined $\zeta(s)$ for $\Re(s) \geq \frac{1}{2}$, $s \neq 1$ we can use this to define $\zeta(s)$ for all $s \neq 0, 1$. Actually, we can use this to define $\zeta(0)$ too. Because $\zeta(1-s)$ has a simple pole at $s = 0$ and $\sin\frac{s\pi}{2}$ has a simple zero at $s = 0$, the right side of 15 is analytic on a neighborhood of 0 except for a removable singularity at $s = 0$. Thus we can define $\zeta(s)$ at the limit of the right side. We thus obtain an analytic continuation of ζ which is defined everywhere except $s = 1$, where it has a simple pole.

The functional equation 15 can be used to derive many important facts about ζ. First, we note that if s is any odd integer > 1, then $\Gamma(1-s)$ has a pole and $2^s\pi^{s-1}\sin\frac{s\pi}{2} \neq 0$. Since $\zeta(s)$ is finite it follows that $\zeta(1-s) = 0$ for all odd integer $s > 1$. Thus ζ has zeros at all the negative even integers. These are called the *trivial zeros* of $\zeta(s)$. In fact, these are all the zeros of ζ with $\Re(s) < 0$. Indeed, Suppose ρ is any zero of ζ such that $\Re(\rho) < 0$. Replacing s with $1-s$ in 15, we get

$$\zeta(1-s) = 2^{1-s}\pi^s \sin\frac{(1-s)\pi}{2}\Gamma(s)\zeta(s).$$

At $s = \rho$, the left side is finite and nonzero so the right side must be as well. Since $\zeta(\rho) = 0$, this means Γ must have a pole at ρ, so $\rho = -1, -2, \ldots$. However, if ρ is odd then $\sin\frac{(1-s)\pi}{2}$ has a zero at ρ which makes the right side 0. Thus, ρ is a negative even integer. This shows that $\zeta(s)$ has no nontrivial zeros with $\Re(s) < 0$. Looking at the functional equation again, we see that this implies all nontrivial zeros of $\zeta(s)$ lie in the strip $0 \le \Re(s) \le 1$. Lastly, we note that the functional equation also implies that if ρ is a nontrivial root of $\zeta(s)$, then $1-\rho$ is also a nontrivial root.

4. Product Representations of the Riemann Zeta Function

To relate the Riemann zeta function to the prime numbers, we use the unique factorization of integers to write, for $\Re(s) > 1$,

$$\sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_{p \text{ prime}}\left(1 + \frac{1}{p^2} + \frac{1}{p^{2s}} + \cdots\right),$$

because if $n = p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r}$, the term $\frac{1}{n^s} = \frac{1}{p_1^{e_1 s}}\cdots\frac{1}{p_r^{e_r s}}$ occurs exactly once in the expansion of the right side. Thus, we have

$$\zeta(s) = \prod_{p \text{ prime}}\left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right) = \prod_{p \text{ prime}}\left(1 - \frac{1}{p^s}\right)^{-1},$$

by the sum of a geometric series. Thus, we have the following:

Theorem 16. *(Euler Product of $\zeta(s)$) For all $\Re(s) > 1$,*

$$\zeta(s) = \prod_{p \text{ prime}}\left(1 - \frac{1}{p^s}\right)^{-1}.$$

We now want to relate this to the zeros of the zeta function. The idea is to factor ζ like a polynomial, i.e. into terms like $(1 - s/\rho)$ where ρ is a root of ζ. This almost works, but we need a few corrections because ζ isn't enough "like a polynomial" to be written exactly this way (ζ has a pole at $s = 1$ and grows too fast):

Theorem 17. *There exists some constants a and b such that for all $s \in \mathbb{C}$,*

$$(s-1)\zeta(s) = e^{a+bs}\prod_{\rho}\left(1 - \frac{s}{\rho}\right)e^{s/\rho}\prod_{n=1}^{\infty}\left(1 + \frac{s}{2n}\right)e^{-s/2n},$$

where ρ ranges over the nontrivial zeroes of $\zeta(s)$, and the product is taken in an order such that each root ρ is paired with $1-\rho$.

The reason we need to pair each ρ with $1-\rho$ is that the product is not absolutely convergent; the order of the terms does matter. However, as long as we pair each root ρ with $1-\rho$ then the resulting product is absolutely convergent, so then we don't have to worry about the order.

The proof of this formula was sketched by Riemann [5] and later proved by Hadamard using the Weierstrass factorization theorem. See [3].

5. Explicit Formulas

We are now ready to prove the explicit formula for. First, we using the two product representations of ζ, we get

$$(s-1)\prod_{p \text{ prime}}\left(1-\frac{1}{p^s}\right)^{-1} = e^{a+bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}\prod_{n=1}^{\infty}\left(1+\frac{s}{2n}\right)e^{-s/2n}.$$

We now take the log of both sides and then take the derivative. Since $-\log(1-x) = x + x^2/2 + x^3/3 + \cdots$, we have

$$\log\left(\prod_{p \text{ prime}}\left(1-\frac{1}{p^s}\right)^{-1}\right) = \sum_{p \text{ prime}} -\log\left(1-\frac{1}{p^s}\right)$$
$$= \sum_{p \text{ prime}}\sum_{m=1}^{\infty}\frac{1}{mp^{ms}}.$$

Thus, the logarithmic derivative of the left side is

$$\frac{1}{s-1} - \sum_{p \text{ prime}}\sum_{m=1}^{\infty}\frac{\log p}{p^{ms}}.$$

On the right side, we get

$$b + \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{s+2n}-\frac{1}{2n}\right).$$

Equating these and rearranging, we get

$$\sum_{p \text{ prime}}\sum_{m=1}^{\infty}\frac{\log p}{p^{ms}} = \frac{1}{s-1} - b - \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) - \sum_{n=1}^{\infty}\left(\frac{1}{s+2n}-\frac{1}{2n}\right), \tag{18}$$

for all $\Re(s) > 1$. In fact, both sides of this equation are just the logarithmic derivative of $\zeta(s)$, $\frac{\zeta'(s)}{\zeta(s)}$. In particular, plugging in $s = 0$ gives $b + 1 = \frac{\zeta'(0)}{\zeta(0)}$, which can be shown to equal $\log(2\pi)$. Now, the idea is then to apply the Perron integral operator:

$$f \to \lim_{T\to\infty}\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} f(s)\frac{X^s}{s}\,ds,$$

where $\sigma > 1$ and $X > 1$. The integral here is shorthand for the contour integral over the curve $\sigma + it$ for $t \in [-T, T]$. Some technical arguments are needed to ensure that this is valid, but assuming we can do this and swap the integrals and the sums, the left hand side becomes

$$\sum_{p \text{ prime}}\log p\sum_{m=1}^{\infty}\lim_{T\to\infty}\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{1}{s}\left(\frac{X}{p^m}\right)^s ds. \tag{19}$$

To evaluate this, we use the follow formula:

Theorem 20 (Perron's formula)**.**

$$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{X^s}{s}\,ds = \begin{cases}1 & \textit{if } X > 1,\\ 0 & \textit{if } 0 < X < 1.\end{cases}$$

The idea behind the proof is the following. First suppose $X > 1$. Let C_T denote the curve from $\sigma + iT$ to $\sigma - iT$ along the left part of the circle with diameter on $[\sigma - iT, \sigma + iT]$. Then, if we let Γ denote the simple closed, positively oriented contour obtained by connecting C_T to $[\sigma - iT, \sigma + iT]$, we obtain

$$\int_\Gamma \frac{X^s}{s}\, ds = \int_{C_T} \frac{X^s}{s}\, ds + \int_{\sigma-iT}^{\sigma+iT} \frac{X^s}{s}\, ds.$$

By the residue theorem (Theorem 14), the integral on the left is just $2\pi i$. Thus,

$$\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \frac{X^s}{s}\, ds = 1 - \frac{1}{2\pi i}\int_{C_T} \frac{X^s}{s}\, ds.$$

As we take T larger, we expect the integral on the right to go to zero, because X^s becomes small when $\Re(s)$ is very small. Thus,

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \frac{X^s}{s}\, ds = 1.$$

When $0 < X < 1$, we use a similar argument. However, the integral $\int_{C_t} \frac{X^s}{s}\, ds$ will not go to zero if we define C_T like before. Thus, we instead let C_T be the right part of the circle with diameter on $[\sigma - iT, \sigma + iT]$. This time, the singularity at 0 is not inside the closed contour Γ, so the residue theorem gives

$$\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \frac{X^s}{s}\, ds = -\frac{1}{2\pi i}\int_{C_T} \frac{X^s}{s}\, ds.$$

Taking limits, we obtain

$$\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \frac{X^s}{s}\, ds = 0.$$

Now, using this, we see that (19) is equal to

$$\sum_{p \text{ prime}} \log p \sum_{p^m < X} 1 = \sum_{p^m < X} \log p = \psi(x),$$

where $\psi(x)$ is the Chebyshev function. Now consider the right side of (18). Applying the Perron operator, and assuming we can interchange the sums and integrals, we get

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \frac{X^s}{s(s-1)}\, ds - b - \sum_\rho \lim_{T\to\infty} \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \frac{X^s}{s(s-\rho)}\, ds + \frac{1}{\rho}$$
$$- \sum_{n=1}^{\infty} \lim_{T\to\infty} \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \frac{X^s}{s(s+2n)}\, ds - \frac{1}{2n}.$$

To evaluate these remaining we do something similar to the explanation of Theorem 20. Each of the integrals is of the form

$$\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \frac{X^s}{s(s-a)},$$

where $\Re(a) \le 1$. Turning this into an integral over a closed contour and using the residue theorem, we get

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \frac{X^s}{s(s-a)} = -\frac{1}{a} + X^a,$$

because $\sigma > 1$, assuming the integral over the added curve goes to 0. Plugging this into the equation, we get

$$\begin{aligned}\psi(x) &= X - 1 - b - \sum_{\rho}\left(\frac{X^\rho}{\rho} - \frac{1}{\rho} - \frac{1}{\rho}\right) - \sum_{n=1}^{\infty}\left(\frac{X^{-2n}}{-2n} + \frac{1}{2n} - \frac{1}{2n}\right)\\ &= X - (b+1) - \sum_{\rho}\frac{X^\rho}{\rho} - \frac{1}{2}\log(1 - X^{-2}),\end{aligned}$$

where we used the Taylor series for $-\log(1-x)$. As we noted before, $b+1 = \frac{\zeta'(0)}{\zeta(0)} = \log(2\pi)$. Thus, we have the following:

Theorem 21 (Riemann-von Mangoldt Explicit Formula)**.** *For any $X > 1$, we have*

$$\psi(X) = X - \sum_{\rho}\frac{X^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log(1 - X^{-2}),$$

where ρ ranges over the non-trivial zeros of ζ.

Remember we need to pair each root ρ with $1-\rho$ for absolute convergence in the sum. This formula is the a reformulation by von Mangoldt of Riemann's original explicit formula. While this formula is perhaps more natural, Riemann's original formula is more directly related to the prime counting function $\pi(x)$. Riemann's original formula is stated in terms of his prime counting function $J(x)$:

Definition 22. We define the function $J(x)$ by the formula

$$J(x) = \sum_{n\geq 1}\sum_{p^n \leq x}\frac{1}{n}.$$

This function can be directly related to the prime counting function $\pi(x)$ by Möbius inversion:

Theorem 23. *We have*

$$J(x) = \sum_{n\geq 1}\frac{\pi(x^{1/n})}{n}$$

and hence by Möbius inversion,

$$\pi(x) = \sum_{n\geq 1}\frac{\mu(n)}{n}J(x^{1/n}).$$

Now we can state Riemann's original formula:

Theorem 24 (Riemann's Original Explicit Formula)**.** *For all $x > 1$, we have*

$$J(x) = \mathrm{li}(x) - \sum_{\rho}\mathrm{li}(x^\rho) - \log 2 + \int_x^\infty \frac{1}{t(t^2-1)\log t}\,dt,$$

where the sum is over all nontrivial zeros of ζ, and

$$\mathrm{li}(x) = \int_0^x \frac{1}{\log t}\,dt.$$

This along with Theorem 23 gives an explicit formula for $\pi(x)$ in terms of the zeros of the zeta function. When von Mangoldt originally proved Thereom 21, he also showed that it is equivalent to this. See [4] or [2].

6. Asymptotic Behavior of $\psi(x)$ and the Riemann Hypothesis

The prime number theorem, which states that $\pi(x) \sim \frac{x}{\log x}$ is well known to be equivalent to the statement $\psi(x) \sim x$. Using theorem 21, it is easy to see why this should be true. Dividing by x and taking a limit, we find that

$$\lim_{x\to\infty} \frac{\psi(x)}{x} = 1 - \lim_{x\to\infty} \sum_{\rho} \frac{x^{\rho-1}}{\rho},$$

because the other terms go to 0. Each term $\frac{x^{\rho-1}}{\rho}$ goes to zero if and only if $\Re(\rho) < 1$. Thus, assuming we can interchange the limit and the sum, the prime number theorem is equivalent to the statement that $\zeta(s) \neq 0$ when $\Re(s) = 1$. Indeed, this is true and is not too difficult to show. Thus, we have the following:

Theorem 25 (The Prime Number Theorem)**.**

$$\psi(x) \sim x$$

Unfortunately, it is not easy to justify the interchanging of the sum and limit in the previous argument due to the fragility of the convergence of the sum. For a rigour proof using more delicate arguments in complex analysis, and a proof of the non-vanishing of $\zeta(s)$ on the line $\Re(s) = 1$, see [2].

In Riemann's original paper, he made a now famous conjecture which is significantly stronger than the non-vanishing of $\zeta(s)$ on the line $\Re(s) = 1$:

Conjecture 26 (Riemann Hypothesis)**.** *If* $\zeta(s) = 0$ *and* $s \neq -2, -4, -6, \ldots$, *then* $\Re(s) = \frac{1}{2}$.

From the explicit formula and the previous discussion, it is no surprise that if true, this would reveal an incredible amount about the distribution of the primes. Since the statement that $\zeta(s)$ has no zeros with real part 1 is equivalent to the prime number theorem, $\pi(x) \sim \frac{x}{\log x}$, then we should expect that determining the exact real part of every zero of the zeta function should give a much stronger bound. Indeed, the Riemann Hypothesis has an equivalent form in terms of the asymptotic of $\pi(x)$.

First, we note that the prime number theorem is equivalent to the statement $\pi(x) \sim \mathrm{Li}(x)$, where

$$\mathrm{Li}(x) = \int_2^x \frac{1}{\log t}\, dt.$$

This follows from the fact that $\mathrm{Li}(x) \sim \frac{x}{\log x}$, which can by shown by integration by parts. This formulation of the prime number theorem is often considered more natural. We can interpret it as saying that the density of primes of size x is about $\frac{1}{\log x}$, so the number of primes less than or equal to x is the integral over this density, i.e.

$$\pi(x) \sim \int_2^x \frac{1}{\log t}\, dt = \mathrm{Li}(x).$$

Assuming the Riemann Hypothesis, we can get a much better bound on how close $\pi(x)$ is to $\mathrm{Li}(x)$. In fact, it turns out that this bound is actually equivalent to the Riemann Hypothesis:

Theorem 27. *The Riemann Hypothesis is equivalent to the bound*

$$\pi(x) = \mathrm{Li}(x) + O(\sqrt{x}\log(x)).$$

For a proof, see [2].

References

[1] J. Bonnar. *The Gamma Function*. CreateSpace Independent Publishing Platform, 2014. ISBN: 9781497387027. URL: https://books.google.com/books?id=2YE0oAEACAAJ.

[2] H. M. Edwards. *Riemann's Zeta Function*. Dover Publications, New York, 2001.

[3] J. Hadamard. "Etude sur les Propri´et´es des Fonctions Enti'eres et en Particulier d'une ´ Fonction Consid´er´ee par Riemann". In: *J. Math. Pures Appl.* (1893).

[4] H. von Mangoldt. "Zu Riemanns Abhandlung "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse"." In: *Journal für die reine und angewandte Mathematik* 114 (1895), pp. 255–305. URL: http://eudml.org/doc/148949.

[5] Bernhard Riemann. "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse". In: *Bernard Riemann's gesammelte mathematische Werke und wissenschaftlicher Nachlass*. Ed. by Richard Dedekind and Heinrich MartinEditors Weber. Cambridge Library Collection - Mathematics. Cambridge University Press, 2013, pp. 136–144.

[6] E.B. Saff and A.D. Snider. *Fundamentals of Complex Analysis with Applications to Engineering and Science*. Prentice Hall, 2003. ISBN: 9780139078743. URL: https://books.google.com/books?id=fVsZAQAAIAAJ.