DILOGARITHM AND HYPERBOLIC TETRAHEDRA

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ABSTRACT. In this paper, we explore properites of the polylogarithm function $Li_k(z)$, focusing mainly on the dilogarithm(k=2) and its extention the Bloch-Wigner function D(z). We start with some computational properties of $li_2(z)$ and some basic values and then introduce D(z) and use it to find the volume of an ideal tetrahedron in $\mathbb{H}^3$

1. Introduction

The polylogarithm is a somewhat natural extension of the taylor series for ln(1-x) A ploylogarithm is defined by the series

$$Li_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$$

which converges for $|z| < 1$ and $m \in \mathbb{Z} > 0$. It has many applications in number theory, geometry and is a key component in many computations in quantum mechanics.

2. Values and equations

The simp1lest one and the one we will focus on is the dilogorithm: $Li_2(z)$ which has relatively few easily computable values:

$$Li_2(0) = 0$$

$$Li_2(1) = \zeta(2) = \frac{\pi^2}{6}$$

$$Li_2(-1) = -\frac{\pi^2}{12}$$

$$Li_2(\frac{1}{2}) = \frac{\pi^2}{12} - \frac{1}{2}\log^2(2) Li_2(-\phi) = -\frac{\pi^2}{10} + \frac{1}{2}\log^2(\phi)$$

There are 8 known in total and the other values are variations of $\frac{1 \pm \sqrt{5}}{2}$. Despite this, there are a lot of functional equations which $Li_2(z)$ satisfies most of which come from the inversion formulas.
To show these formulas to be true it will be helpful to consider another form of the dilogarithm.

$$Li_2(z) = -\int_0^z \frac{ln(1-u)}{u} du \tag{1}$$

Which converges to Li_2 on $\mathbb{C} \setminus [1, \infty)$ Which can be checked to be equivalent to our original definition be a simple swapping of sums. Now the inversion formulas.

Proposition 2.1.

$$Li_2(\frac{1}{z}) = -Li_2(z) - \frac{\pi^2}{6} - \frac{1}{2}\log^2(-z)$$
$$Li_2(\frac{1}{z}) = -Li_2(z) + \frac{\pi^2}{6} - \log(z)\log^2(1-z)$$

The proof for both is pretty similar and somewhat boring so we'll just do an outline of $\frac{1}{z}$ case.

Proof. We will show that $Li_2(z) + Li_2(\frac{1}{z}) = -\frac{1}{2}log^2(-z) - \frac{\pi^2}{6}$ and to do this we consider the integral representation of $Li_(- z)$ and $Li_2(-\frac{1}{z})$ which, after a sub of$u = -z$ we get

$$\int_0^z \frac{\log(1+u)}{u}du \int_0^{\frac{1}{z}} \frac{log(1+u)}{u}$$

Then, we substitute x = $1/u$ and simplify to get

$$\int_0^\infty \frac{log(1+y)}{y}dy + \lim_{y\to\infty}\frac{1}{2}ln^2 y + \frac{1}{2}log^2(y)$$

If we split it up to the part from 0 to 1 and the part from 1 to ∞, then substitute $y = \frac{1}{u}$ into the inifinite integral we get

$$-\frac{\pi^2}{6} - \lim_{y\to 0}\frac{1}{2}ln^2 y + lim_{y\to\infty}\frac{1}{2}ln^2 y + \frac{1}{2}log^2(y)$$

which simplifies down to our desired result. □

From the previous two equations we get that $Li_2(z), Li_2(\frac{1}{1-z}, Li_2(\frac{z-1}{z}), Li_2(\frac{1}{z}), Li_2(1-z), Li_2(\frac{z}{z-1}))$ are equal modulo elementary functions(i.e$log^k(x)$, etc. . .)

3. Multilpe Zeta values and Multiple logarithms

One of the many ways that multiple polylogarithms values can be used is to evaluate multiple zeta values, which are objects which have garnered a lot of attention as they seem to show up very often in expressions for Feynman amplitudes in quantum field theory.

$$Li_{x_1,\ldots,x_m}(k_1,\ldots,k_m) = \sum_{n_1>\ldots n_i\geq 1}\frac{x_1^{n_1}\ldots x_m^{n_m}}{n^{k_1}\cdots n^{k_m}}$$

. At $x_1 = x_2 \ldots x_i = 1$ the logarithm becomes a multiple zeta value $\zeta(k_1, k_2 \ldots k_i)$ One of the main invariants between zeta values is their weight, $\sum_{n\leq i} k_i$, so there is an algebraic relation between Li_2 and $Li_{x,y}(1,1)(\sum_{0<n<m}\frac{x^n y^m}{nm})$ given as follows:

Proposition 3.1.

$$Li_{x,y}(1,1) = Li_2(\frac{xy-y}{1-y}) - Li_2(\frac{-y}{1-y}) - Li_2(xy)$$

Which we will prove by taking the derivative and taking initial values.

Proof. If we take

$$\frac{\delta}{\delta y}Li_2(1,1) = \sum_{n>m>0}\frac{x^m}{m}y^{n-1} = \sum_{m=1}^{\infty}\frac{x^m}{m}\frac{y^m}{1-y} = \frac{1}{1-y}log\frac{1}{1-xy}$$

If we calculate the derivative on the right hand side we get the same expression and since both vanish at y = 0, the equation holds for all y and analogously all x. □

The next results will be stated but not prove, but both these results were already known to Euler, the first one from his book "Meditations about a singular type of series" gives an attempt at a closed for for the ever elusive $\zeta(3)$ by considering another multiple polylogarithm of the same weight.

$$\zeta(3) = \zeta(2,1)$$

The other equation gives a way to relate muliples of zeta values to multiple zeta values.

$$\zeta(s_1)(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2)$$

Further discussions of multiple logarithms and multiple zeta vlaues are in [GF17]

4. Bloch-Wigner and Hyperbolic Volume

However, this definition of $Li_2(z)$ jumps by $2\pi log|z|$. So to counteract this jump we add on a term of $arg(1-z)log|z|$ which takes on values between $-\pi$ and π and thus the new function $Li_2(z) + arg(1-z)log|z|$. Its imaginary part

$$D(z) = \Im(Li_2(z) + arg(1-z)log|z|)$$

If we consider the function $\tilde{D}(z_0, z_1, z_2, z_3) = D(\frac{z0-z_2}{z_0-z_3}\frac{z_1-z_3}{z_1-z_2})$ This function, called the Bloch-Wigner function has some interesting properties. It is an analytic extension of the imaginary part of Li_2. To be more specific, D(z) is continuous on all of $\mathbb{C}$ and furthermore it is analytic on $P^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ One of the main advantages to using D(z) is that equivelance between $D(z), D(\frac{1}{z})$ and $D(1-z)$ loses the elementary terms and we have the formulation $D(\bar{z}) = -D(z)$ Another property that comes from this function is that all the One of which is that it can be interpreted as the volume of a hyperbolic ideal tetrahedron. (i.e a tetrahedron in with all vertices in $\mathbb{P}^1(\mathbb{C})$). However we can preform actions from $SL_2(\mathbb{C})$, which are isometries on hyperbolic space, to move three of the points to $0, 1, \infty$ and z to simply the volume to $D(z)$. Tetrahedra are the foundation of manifolds and so understanding them allows us to better understand 3- manifolds in general.

To prove this, following [Mil82] we will consider a new function $\lambda(\theta)$, which is a slight modification of the Lobachevsky function, but we will still keep the name, we will then calculate the volume for a specific type of simplex and show that the volume of our tetrahedron can be written as a sum of the volumes of these simplices. First, the lobochevsky function:

$$\lambda(\theta) = -\int_0^\theta log|2sin(z)|dz$$

Proposition 4.1. $\Im Li_2(e^{2i\theta}) = 2\lambda(\theta)$

Proof. Consider the integral form of the dilogarithm as in (1) and make the substitution u $= e^{2i\theta}$ for when $|u| \leq 1$. The integrand then becomes

$$\pi - 2\theta - 2\pi i log(2\sin(\theta))d\theta$$

for $0 < \theta < \pi$ Then, if we integrate from 0 to θ we get that

$$Li_2(e^{2i\theta}) - Li_2(1) = \theta(\pi - \theta) + 2i\lambda(\theta)$$

If we consider only the imaginary parts of both sides we have that $\Im Li_2(e^{2i\theta}) = 2\lambda(\theta)$ □

In this way, the Lobochevsky function is a "polarization" of our dilogarithm function. We will find the volume of our tetrahedron in terms of angles and then "unpolarize" the function from Lobochevsky to get it in terms of the dilogarithm. First, though, there are two more property of the Lobochevsky function that we will need.

Proposition 4.2. *The Lobochevsky function is periodic with period π and furthermore it's an odd function.*

Proof. Consider $\lambda'(\theta) = -2\log|\sin(\theta)|$. The function clearly has a period of π so now all we need to show is that two points separated by period π are the same. $\lambda(0)$ is clearly 0 and by a simple calculation, one could check that $\lambda(\pi) = 0$. The function is clearly odd since $\int_0^a = -\int_0^{-a}$ □

The next property gives us a way to simplify the theta values within a function.

Proposition 4.3.

$$\lambda(n\theta) = \sum_{j \mod n} n(\lambda(\theta + j\pi/n))$$

Where we sum over all residue classes of n.

Proof. Consider the equation $z^n - 1 = \prod_{j=0}^{n-1} z - e^{2i\pi j/n}$. If we take z to be of the form $e^{2\pi i u}$ and consider it's imaginary component we get

$$2sinnu = \prod_{j=0}^{n-1} 2\sin(u + j\pi/n)$$

Taking the logarithm, integrating and then multiplying both sides by n we get our desired result. □

For example let's consider $\lambda(2\theta)$

$$\lambda(2\theta) = \lambda(\theta) + \lambda(\theta + \pi/2)$$

Now we can go on to finding volumes. We will be using the upper half-plane model of hyperbolic space for this called $\mathbb{H}^3$ To find the volume we will need to devise a metric for our space. For this, it is useful to at least get some visualization of hyperbolic half plane.
In $\mathbb{H}^3$.Every hemisphere of radius of r from the origin corresponds to a plane of x, y, r in euclidean space. In this model, the metric ds satisfies, $ds^2 = \frac{dx^2+dy^2+dz^2}{z^2}$ and a change in volume $dV = \frac{dxdydz}{z^3}$ First consider the tetrahedron with three right angle and only one point at infinity and the other three points lie on the unit sphere.The three right angles restrictions is fine because any tetrahedron can be subdivided into such objects, as we shall see. This means that we have dihedral angles between the vertical planes of $\alpha, \pi/2 - \alpha, \gamma$. Now we'll project the points that are on the unit sphere down to the half plane. The angles on the projected half rectangle will have angles equal to $\alpha, \pi/2 - \alpha, \gamma$.To find this tetrahedrons volume, we can integrate over z and, for the triangle, we can parameterize the triangle by letting $0 \geq x \geq \gamma$ and $0 \geq y \geq xtan\alpha$

$$V = \int\int_{x,y\in T}\int_{z\geq\sqrt{1-x^2-y^2}} \frac{dxdydz}{z^3}$$
$$= \int\int_T \frac{dxdy}{2(1-x^2-y^2)}$$

Now if we set a = $\sqrt{1-x^2}$ and we take the bounds of the triangle integral to be what we defined before we get

$$V = \int_0^{cos\gamma}\int_0^{x\tan\alpha} \frac{dxdy}{2(a^2-y^2)} = \int_0^{cos\gamma} \frac{x}{4a}\log\frac{a+x\tan\alpha}{a-xtan\alpha} = \int_0^{cos\gamma}\frac{dx}{4a}\log\frac{acos\alpha + x\sin\alpha}{acos\alpha - xsin\alpha}$$

If we set x = $sin\theta$, then a =$\cos\theta$ and $dx = -ad\theta$ Making this substitution into the previous equation we get that

$$V = -\frac{1}{4}\int_{\pi/2}^{\gamma}\log(\frac{2sin(\theta+a)}{2sin(\theta-a)})d\theta = -\frac{1}{4}\int_{\pi/2}^{\gamma}\log(2sin(\theta+a)-2sin(\theta-a))d\theta \tag{2}$$

Which lends us to, since, by 4.2 $\lambda(\pi/2-\alpha) = -\lambda(\pi/2+\alpha)$

$$\frac{1}{4}\lambda(\gamma+\alpha) - \lambda(\gamma-\alpha) + 2\lambda(\pi/2-\alpha)$$

Extending another point to infinity, which can be visualized as moving a point to the intersection between the plane and the unit sphere, we get that $\alpha = \gamma$. Leading to the simplifaction:

$$\frac{1}{4}\lambda(2\alpha) + 2\lambda(\pi/2-\alpha)$$

by the Lobochevsky double angle formula:

$$\lambda(2\alpha) = \lambda(\alpha) + \lambda(\alpha+\pi/2)$$

and by letting $\lambda(\pi/2-\alpha) = -\lambda(\pi/2+\alpha)$ we can further simplify the volume of our tetrahedron to just

$$\frac{1}{2}\lambda(\alpha)$$

. Now, to see how an arbitrary ideal simplex decomposes into the aforementioned ones.
First, consider a simplex in $\mathbb{H}^3$ and move one point to infinity and put the base all on the unit sphere. Then drop a perpendicular line l from the point at infinity down to the base at point x and draw lines from x to be perpendicular lines to the edges of the base and then connect x to the vertices. Projecting from the unit sphere to the (x, y) gives a triangle inscribed in a circle with the point x at the center. Then a quick check of the angles shows that we have the simplices with three right angles and two points at infinity.
With this, we have a subdivision of our ideal tetrahedron with dihedral angles α, β, γ into two 6 tetrahedra consisting of pairs of tetrahedrons with the afore mentioned angle as their discernment giving us that

$$\mathbf{V} = 2(1/2\lambda(\alpha) + 1/2\lambda(\beta) + 1/2\lambda(\gamma)) \tag{3}$$

One thing to note is that for $z = e^{2i\theta}$ D(z) = $\Im(Li_2(z))$ since $|z| = 1$. From this we get that

$$\mathbf{V} = D(z_1) + D(z_2) + d(z_3)$$

how this can be converted to $\tilde{D}(z_1, z_2, z_3, z_4)$ is a bt outside the scope of this paper. For discussions of $D_k(z)$ the generilized version of D(z) see [Zag07]

References

[GF17] JI Burgos Gil and Javier Fresán. Multiple zeta values: from numbers to motives. *Clay Mathematics Proceedings, to appear*, 379, 2017.

[Mil82] John W. Milnor. Hyperbolic geometry: The first 150 years. *Bulletin of the American Mathematical Society*, 6(1):9–24, 1982.

[Zag07] Don Zagier. The dilogarithm function. In *Frontiers in Number Theory, Physics, and Geometry II: On Conformal Field Theories, Discrete Groups and Renormalization*, pages 3–65. Springer, 2007.