Apery's Theorem

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1 Introduction

One of the most important analytical functions is $\zeta(s)$. It is defined with the domain of the complex plane and is a special case of Dirichlet Series given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Leonhard Euler first proved that if n is a positive integer, then $\zeta(2n) = \frac{p}{q}\pi^{2n}$, for a rational number $\frac{p}{q}$, which led to the proof of the irrationality of $\zeta(2n)$ due to the fact that all powers of π are irrational.

No formula exists for the values of $\zeta(2n+1)$, so it is still open as to whether or not these values are irrational. However, Roger Apery first outlined a proof in 1978 with a proof that $\zeta(3)$ is irrational. Several more alternative proofs have been given since then.

Though not much more is known for exactly which values of $\zeta(2n+1)$ are irrational, results have been shown on irrationality of subsets of these numbers. For example, it is known there exist infinitely many n such that $\zeta(2n+1)$ is irrational, and at least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is as well.

Apery's Theorem states the claim that $\zeta(3)$ is irrational. We will call upon the work of Alfred van der Poorten to prove Apery's Theorem.

2 Proof

The main part of Apery's original proof relies on the following irrationality criterion.

Definition 2.1 (Dirichlet Irrationaliy Criterion)**.** *If there exists a positive constant δ and sequences of integers $\{p_n\}, \{q_n\}$ such that $\frac{p_n}{q_n} \neq \beta$, and*

$$\left|\beta - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{1+\delta}}$$

for all positive integers n, then β is irrational.

With this in mind, construct the following functions.

Definition 2.2. *Let*

$$b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \qquad a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 c_{n,k},$$

where

$$c_{n,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}}.$$

Then $a_0 = 0$, $a_1 = 6$, $b_0 = 1$, $b_1 = 5$, and each sequence $\{a_n\}, \{b_n\}$ satisfies the recurrence relation given by

$$n^3 u_n + (n-1)^3 u_{n-2} = (34n^3 - 51n^2 + 27n - 5)u_{n-1}.$$

To do the proof, we must show that $\frac{a_n}{b_n}$ approaches $\zeta(3)$ fast enough to conclude the irrationality of $\zeta(3)$, using the Dirichlet irrationality criterion. But first to apply this we must find the denominators of the sequences $\{a_n\}$ and $\{b_n\}$.

We can clearly see from the definition above that b_n is an integer for all n. However, a_n isn't as simple.

Lemma 2.1. *We have that $2(\mathrm{lcm}(1, 2, \cdots n))^3 c_{n,k}\binom{n+k}{k}$ is an integer.*

Proof. Clearly the left term of $c_{n,k}$ is divisible by $\mathrm{lcm}(1, 2, \cdots, n)^3$, so we only need to consider this for the right-side denominator. In order to do this, we will compare the number of times a given prime p divides the denominator divided by $2\binom{n+k}{k}$ to the number of times it divides $\operatorname{lcm}(1,2,\cdots,n)^3$(notice the divided by $2\binom{n+k}{k}$ in the denominator comes from multiplying the entire expression by that value). We will use the notation $\nu_p(x)$ to denote the largest power of p dividing x.

First notice that the value of $\nu_p(\operatorname{lcm}(1,2,\cdots,n)^3) = 3\log_p n$. Next, consider the value of $\nu_p\left(\binom{a}{b}\right)$ for values a and b. To bound this, we will use Legendre's formula.

Definition 2.3 (Legendre's formula)**.** *For any prime number p and positive integer n, we have that* $\nu_p(n!) = \sum_{i\geq 1} \left\lfloor \frac{n}{p^i} \right\rfloor$.

From this, we have that

$$\nu_p\left(\binom{a}{b}\right) = \nu_p\left(\frac{a!}{b!(a-b)!}\right) = \nu_p(a!) - \nu_p(b!) - \nu_p((a-b)!) = \sum_{i\geq 1}\left(\left\lfloor\frac{a}{p^i}\right\rfloor - \left\lfloor\frac{b}{p^i}\right\rfloor - \left\lfloor\frac{a-b}{p^i}\right\rfloor\right).$$

For all $i > \log_p a$, the summand is 0 due to the fact that all the terms are as well. Then for all $i \leq \nu_p(b)$, notice that $\left\lfloor\frac{b}{p^i}\right\rfloor + \left\lfloor\frac{a-b}{p^i}\right\rfloor = \frac{b}{p^i} + \left\lfloor\frac{a-b}{p^i}\right\rfloor = \left\lfloor\frac{a-b+b}{p^i}\right\rfloor = \left\lfloor\frac{a}{p^i}\right\rfloor$. Thus in this case, the summand is also 0. Thus we have that $\nu_p\left(\binom{a}{b}\right) \leq \log_p a - \nu_p(b)$.

Now we will use the identity $\frac{\binom{n+m}{m}}{\binom{n+k}{k}} = \frac{\binom{k}{m}}{\binom{n+k}{k-m}}$. We have that

$$\begin{aligned}
\nu_p\left(\frac{m^3\binom{n}{m}\binom{n+m}{m}}{\binom{n+k}{k}}\right) &= \nu_p\left(\frac{m^3\binom{n}{m}\binom{k}{m}}{\binom{n+k}{k-m}}\right) \\
&\leq \nu_p\left(m^3\binom{n}{m}\binom{k}{m}\right) \\
&\leq 3\nu_p(m) + \log_p n + \log_p k - 2\nu_p(m). \\
&= \nu_p(m) + \log_p n + \log_p k.
\end{aligned}$$

Then because $m \leq k \leq n$, we can see that each of those terms is less than or equal to $\log_p n$. So we have shown the number of times an arbitrary prime p divides the denominator of $2c_{n,k}\binom{n+k}{k}$ is less than the number of times it divides $\operatorname{lcm}(1,2,\cdots,n)^3$, thus showing that $2(\operatorname{lcm}(1,2,\cdots n))^3 c_{n,k}\binom{n+k}{k} \in \mathbb{Z}$. $\square$

Therefore since a_k has a factor of $c_{n,k}\binom{n+k}{k}$ in each term, we know it is rational with a denominator dividing $2\text{lcm}(1,2,\cdots,n)^3$.

Now, it is known in the field of number theory that $\log(\text{lcm}(1,2,\cdots,n)) = \sum_{x\leq n}\Lambda(x) = \psi(n)$, where $\Lambda(x)$ is the function that returns $\log p$ when x is a perfect power of p and 0 otherwise. It has also been shown that $\frac{\psi(n)}{n}$ is bounded, meaning that $\psi(n) = O(n)$, and $\text{lcm}(1,2,\cdots,n) = e^{\psi(n)} = O(e^n)$, which is necessary later in the proof.

Next, notice that multiplying the recursive relation for a_n by b_{n-1} we have

$$n^3 a_n b_{n-1} - (34n^3 - 51n^2 + 27n - 5)a_{n-1}b_{n-1} + (n-1)^3 a_{n-2}b_{n-1} = 0.$$

Doing a similar multiplication for the recursive relation for b_n also gives

$$n^3 b_n a_{n-1} - (34n^3 - 51n^2 + 27n - 5)b_{n-1}a_{n-1} + (n-1)^3 b_{n-2}a_{n-1} = 0.$$

Subtracting these two equations, we have that $n^3(a_n b_{n-1} - a_{n-1}b_n) - (n-1)^3(a_{n-1}b_{n-2} - a_{n-2}b_{n-1}) = 0$.

Now, define $F(n) = a_n b_{n-1} - a_{n-1}b_n$. Then $F(1) = 6$, and $F(n) = \dfrac{(n-1)^3}{n^3}F(n-1)$. This recurrence leads us to discover the closed form $F(n) = a_n b_{n-1} - a_{n-1}b_n = \dfrac{6}{n^3}$.

From this, Apery's next claim was that

$$\left|\zeta(3) - \frac{a_n}{b_n}\right| = \sum_{k=n+1}^{\infty} \frac{6}{k^3 b_k b_{k-1}}.$$

To get this, define $\chi_n = \zeta(3) - \dfrac{a_n}{b_n}$. We have that $\chi_\infty = 0$. We also have

$$\chi_k - \chi_{k-1} = \frac{a_{k-1}}{b_{k-1}} - \frac{a_k}{b_k} = \frac{a_k b_{k-1} - a_{k-1}b_k}{b_k b_{k-1}} = -\frac{6}{k^3 b_k b_{k-1}}.$$

Apery's claim follows simply from here by summing this value up from $k = n+1$ to ∞.

Then, since b_k is always increasing, we have that $\left|\zeta(3) - \dfrac{a_n}{b_n}\right| \leq \displaystyle\sum_{k=n+1}^{\infty} \frac{6}{k^3 b_{k-1}^2} \leq b_n^{-2} \cdot 6 \cdot \zeta(3)$, so $\zeta(3) - \dfrac{a_n}{b_n} = O(b_n^{-2})$.

We can also approximate b_n using it's recursive formula. Notice that dividing it by n^3 and getting rid of the negligible terms yields $b_n - 34b_{n-1} + b_{n-2} = 0$, and we have that $(1 \pm \sqrt{2})^4$ are the solutions to this characteristic equation. This leads to the fact that $b_n = O(\alpha^n)$, where $\alpha = (1 + \sqrt{2})^4$.

From this we also have that $\zeta(3) - \dfrac{a_n}{b_n} = O(\alpha^{-2n})$.

Now we are at the closing part of the proof, and wish to apply the Dirichlet irrationality criterion. However, we must note that the sequence $\{a_n\}$ does not consist of integers. To fix this, define the new functions

$$p_n = 2(\mathrm{lcm}(1, 2, \cdots, n))^3 a_n, \qquad q_n = 2(\mathrm{lcm}(1, 2, \cdots, n))^3 b_n,$$

so that we have $p_n, q_n \in \mathbb{Z}$.

Thus $q_n = O(\alpha^n \cdot e^{3n})$, using our bound earlier for $\mathrm{lcm}(1, 2, \cdots, n)$. Finally, we have that

$$\zeta(3) - \frac{p_n}{q_n} = O(\alpha^{-2n}) = O\left(\frac{1}{q_n^{1+\delta}}\right),$$

where $\delta = \dfrac{\log\alpha - 3}{\log\alpha + 3} > 0$. Therefore we have that from the irrationality criterion that $\zeta(3)$ is irrational, completing the proof.

3 Apery's formula for $\zeta(3)$

During his work on proving the irrationality of $\zeta(3)$, Apery also discovered the formula

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}.$$

Though this didn't directly apply to his original proof, it is still a handy formula to work with. We shall now prove this result.

Lemma 3.1. *Given a sequence $a_1, a_2, \cdots, a_r$, we have that*

$$\frac{a_1 a_2 \cdots a_r}{x(x+a_1)\cdots(x+a_r)} + \sum_{i=0}^{r} \frac{a_1 a_2 \cdots a_{i-1}}{(x+a_1)(x+a_2)\cdots(x+a_i)} = \frac{1}{x}$$

Proof. Consider the partial fraction decomposition

$$\frac{1}{x} = \frac{1}{x+a_1} + \frac{a_1}{x(x+a_1)},$$

which is our required result for $r = 1$. We will now show with induction on r that the given formula is true. Assume that

$$f(x) = \frac{a_1 a_2 \cdots a_{r-1}}{x(x+a_1)\cdots(x+a_{r-1})} + \sum_{i=0}^{r-1} \frac{a_1 a_2 \cdots a_{i-1}}{(x+a_1)(x+a_2)\cdots(x+a_i)} = \frac{1}{x}.$$

Then we have that

$$\begin{aligned}
&\frac{a_1 a_2 \cdots a_r}{x(x+a_1)\cdots(x+a_r)} + \sum_{i=0}^{r} \frac{a_1 a_2 \cdots a_{i-1}}{(x+a_1)(x+a_2)\cdots(x+a_i)} \\
&= f(x) + \frac{a_1 a_2 \cdots a_r}{x(x+a_1)\cdots(x+a_r)} + \frac{a_1 a_2 \cdots a_{r-1}}{(x+a_1)\cdots(x+a_r)} - \frac{a_1 a_2 \cdots a_{r-1}}{x(x+a_1)\cdots(x+a_{r-1})} \\
&= \frac{1}{x} + \frac{(a_1 a_2 \cdots a_{r-1})(a_r + x - (x+a_r))}{x(x+a_1)\cdots(x+a_r)} \\
&= \frac{1}{x}.
\end{aligned}$$

Thus the proof is done. □

Now we will utilize this to find a form for $\zeta(3)$.

Substitute $x = n^2$, $a_i = -k^2$, and $r = n-1$ into Lemma 3.1 such that we have

$$\sum_{i=0}^{n-1} \frac{(-1)^{i-1}(k-1)!^2}{(n^2-1^2)(n^2-2^2)\cdots(n^2-k^2)} = \frac{1}{n^2} - \frac{(-1)^{n-1}(n-1)!^2}{n^2(n^2-1^2)\cdots(n^2-(n-1)^2)}.$$

Notice that

$$\frac{(n^2-1^2)\cdots(n^2-(n-1)^2)}{(n-1)!^2} = \frac{1\cdot 2\cdots(n-1)\cdot(n+1)\cdots(2n-1)}{(n-1)!^2} = \frac{(2n)!}{2\cdot n!^2} = \frac{\binom{2n}{n}}{2}.$$

Substituting this back in yields that our last expression is equal to $\frac{1}{n^2} - \frac{2(-1)^{n-1}}{n^2\binom{2n}{n}}$. Following along the footsteps of Apery's proof, our next step is to define the function

$$\epsilon_{n,k} = \frac{1}{2}\cdot\frac{k!^2(n-k)!}{k^3(n+k)!},$$

as this satisfies the relation $(-1)^k n(\epsilon_{n,k}-\epsilon_{n-1,k}) = \frac{(-1)^{i-1}(k-1)!^2}{(n^2-1^2)(n^2-2^2)\cdots(n^2-k^2)}$.

Then we have that

$$\begin{aligned}
\sum_{n=1}^{N}\sum_{i=1}^{n-1}(-1)^k(\epsilon_{n,k}-\epsilon_{n-1,k}) &= \sum_{n=1}^{N}\frac{1}{n^3} - 2\sum_{n=1}^{N}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}} \\
&= \sum_{k=1}^{N}(-1)^k(\epsilon_{N,k}-\epsilon_{k,k}) \\
&= \sum_{k=1}^{N}\frac{(-1)^k}{2k^3\binom{N+k}{k}\binom{N}{k}} + \frac{1}{2}\sum_{k=1}^{N}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}}.
\end{aligned}$$

Thus we have that $\sum_{n=1}^{N}\frac{1}{n^3} - 2\sum_{n=1}^{N}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}} = \sum_{k=1}^{N}\frac{(-1)^k}{2k^3\binom{N+k}{k}\binom{N}{k}} + \frac{1}{2}\sum_{k=1}^{N}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}}$. Taking the limit as N approaches infinity makes the first term on the right approach 0, and the first term on the left becomes $\zeta(3)$. We can then move all terms other than $\zeta(3)$ to the right and change our summation variables which yields

$$\boxed{\zeta(3) = \frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.}$$

4 An Alternative Proof by Frits Beukers

In 1979, Frits Beukers found an alternative proof of Apery's Theorem using integrals involving the shifted Legendre Polynomials $\tilde{P_n}(x)$. Roughly speaking, Beukers showed that

$$\int_0^1 \int_0^1 \frac{-\log(xy)}{1-xy} \tilde{P_n}(x)\tilde{P_n}(y)\, dx\, dy = \frac{A_n + B_n\zeta(3)}{\text{lcm}(1, 2\cdots, n)^3}$$

for integer sequences A_n and B_n. Then assuming that $\zeta(3)$ was rational with denominator b, he was able to show that $|A_n + B_n\zeta(3)| < \frac{1}{b}$ for sufficiently large n, contradicting the fact it was rational.

5 More on the ζ function

In 1737, Euler was able to connect the Zeta function to prime numbers by using the Euler Product identity

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}.$$

Notice that at $s = 1$, the left side diverges, while the right side is the product of all values of $\frac{p-1}{p}$. If you take the natural logarithm of both sides, since $\log\left(\frac{p-1}{p}\right) \approx \frac{1}{p}$, this can be used to show that the sum of the reciprocals of all primes diverges.

There have also been many found analytical continuations of the zeta function. Using the usual summation definition, it is true that $\zeta(s)$ only converges for s with $\Re(s) > 1$. These continuations are alternate forms of $\zeta(s)$ with larger ranges of convergence. For example, the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s),$$

where $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\, dt$, can be used to extend the range of convergence of $\zeta(s)$ to $\mathbb{C}\backslash\{1\}$. At $s = 1$, the sum is the harmonic series which diverges to $+\infty$.

One of the most famous unsolved problems in mathematics is the Reimann hypothesis, conjecturing that all non-trivial zeros of $\zeta(s)$ satisfy $\Re(s) = \frac{1}{2}$(We consider the trivial zeros to be all negative even integers).