DIVISORS OF $n!$

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Abstract.
We look closely at Pomerance, Erdös, Ivíc, and Graham's work on the number of divisors of $n!$, detailed in [EGIP96].

1. Asymptotics for $d(n!)$

We start by constructing an asymptotic for $d(n!)$ in terms of elementary functions. First, we will need a slightly more powerful version of the Prime Number Theorem at various points in this paper, which we will state but not prove.

Theorem 1.1 (Prime Number Theorem)**.** *Let $\pi(x)$ be the number of primes $\leq x$. We have*

$$\pi(x) = \mathrm{Li}(x) + O(xe^{-a\sqrt{\log x}})$$

for some positive constant a.

A proof of this fact is provided in Chapter 18 of [Dav80]. We now introduce our first asymptotic for $d(n!)$.

Theorem 1.2. *Let*

$$c_k = \int_1^\infty \frac{\log\left(\lfloor t \rfloor + 1\right)}{t^2} \log^k(t)\, dt.$$

Then for any integer $K \geq 0$,

$$d(n!) = \exp\left(\frac{n}{\log n} \sum_{k=0}^{K} \frac{c_k}{\log^k n} + O\left(\frac{n}{\log^{K+2} n}\right)\right).$$

Proof. A classical result due to Legendre gives us that

$$n! = \prod_{p \text{ prime}} p^{w_p(n)},$$

where

$$w_p(n) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

Thus

$$\log d(n!) = \log\left(\prod_{p \text{ prime}} (w_p(n) + 1)\right) = \sum_{p \text{ prime}} \log(w_p(n) + 1).$$

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We can split this sum into primes $p \le n^{3/4}$ and primes $n^{3/4} < p \le n$. Noting that $w_p(n) < n\sum_{k=1}^{\infty} p^{-k} = \frac{n}{p-1}$, we have

$$\sum_{\substack{p \le n^{3/4} \\ p \text{ prime}}} \log(w_p(n)+1) < \sum_{\substack{p \le n^{3/4} \\ p \text{ prime}}} \log\left(\frac{n}{p-1}+1\right).$$

We know that $\log\left(\frac{n}{p-1}+1\right) = O(\log n)$, so

$$\sum_{\substack{p \le n^{3/4} \\ p \text{ prime}}} \log(w_p(n)+1) = O(\log(n)\pi(n^{3/4})) = O(n^{3/4})$$

by the Prime Number Theorem. For the second part of the sum, we note that $w_p(n) = \left\lfloor \frac{n}{p} \right\rfloor$ (as all the other terms in the sum are zero), so

$$\sum_{\substack{n^{3/4} < p \le n \\ p \text{ prime}}} \log(w_p(n)+1) = \sum_{\substack{n^{3/4} < p \le n \\ p \text{ prime}}} \log\left(\lfloor n/p \rfloor + 1\right).$$

Rewriting this as a Riemann-Stieltjes integral gives us

$$\sum_{\substack{n^{3/4} < p \le n \\ p \text{ prime}}} \log\left(\lfloor n/p \rfloor + 1\right) = \int_{n^{3/4}}^{n} \log\left(\lfloor n/x \rfloor + 1\right)\, d\pi(x).$$

We can use the approximation from Theorem 1.1 to find

$$\int_{n^{3/4}}^{n} \log\left(\lfloor n/x \rfloor + 1\right)\, d\pi(x) = \int_{n^{3/4}}^{n} \frac{\log\left(\lfloor n/x \rfloor + 1\right)}{\log x}\, dx + \int_{n^{3/4}}^{n} \log\left(\lfloor n/x \rfloor + 1\right)\, dR(x)$$

where $R(x) = O\left(xe^{-a\sqrt{\log x}}\right)$. The second integral has smaller average order than the first, so we need to find an asymptotic for the integral

$$\int_{n^{3/4}}^{n} \frac{\log\left(\lfloor n/x \rfloor + 1\right)}{\log x}\, dx = n\int_{1}^{n^{1/4}} \frac{\log\left(\lfloor t \rfloor + 1\right)}{t^2 \log(n/t)}\, dt.$$

Using the fact that $\log(n/t) = \log(n) - \log(t)$, we have

$$\begin{aligned} \frac{1}{\log(n/t)} &= \frac{1}{\log(n) - \log(t)} \\ &= \frac{1}{\log n}\left(\frac{1}{1 - \frac{\log t}{\log n}}\right) \\ &= \frac{1}{\log n}\sum_{k=0}^{\infty} \frac{\log^k t}{\log^k n}. \end{aligned}$$

Truncating this at $k = K$ gives us

$$\int_{n^{3/4}}^{n} \frac{\log\left(\lfloor n/x \rfloor + 1\right)}{\log x}\, dx = \frac{n}{\log n}\sum_{k=0}^{K} \frac{1}{\log^k n}\int_{n^{1/4}}^{n} \frac{\log\left(\lfloor t \rfloor + 1\right)}{t^2} \log^k t\, dt + O\left(\frac{n}{\log^{K+2} n}\right)$$

for any fixed integer K. Because the error remains the same order of magnitude, we can extend our integral upper bound to ∞ and lower bound to 1. Using c_k to represent this integral, we have

$$d(n!) = \exp\left(\frac{n}{\log n}\sum_{k=0}^{K}\frac{c_k}{\log^k n} + O\left(\frac{n}{\log^{K+2} n}\right)\right).$$

■

We can evaluate the integral at $K = 0$ for the least strict bound computable using this formula. We find that

$$c_0 = \int_1^\infty \frac{\log\left(\lfloor t\rfloor + 1\right)}{t^2}\,dt = \sum_{k=2}^{\infty}\int_{k-1}^{k}\frac{\log k}{t^2}\,dt = \sum_{k=2}^{\infty}\log k\left(\frac{1}{k} - \frac{1}{k-1}\right) \approx 1.25775,$$

so that

$$d(n!) = \frac{c_0 n}{\log n} + O\left(\frac{n}{\log^2 n}\right).$$

2. Asymptotics for $d(n!)/d((n-1)!)$

We first provide a few definitions that will be helpful in proving the theorems in this section.

Definition 2.1. We define $S(n)$ to be the sum of the prime divisors of n (counted with multiplicity).

For instance, because $45 = 3^2 \cdot 5$, we have $S(45) = 3 \cdot 2 + 5 = 11$.

Definition 2.2. We define $\Omega(n)$ to be the number of prime divisors of n (counted with multiplicity).

Using 45 as an example again, we have $\Omega(45) = 3$.

Lemma 2.3.

$$1 + \frac{S(n)}{2n} \le \frac{d(n!)}{d((n-1)!)} \le 1 + \frac{2S(n)}{n}.$$

Proof. We will write $p^a \mid\mid n$ to indicate that p^a is the largest power of p dividing n. We can first prove the upper bound. Note that

$$\begin{aligned}\frac{d(n!)}{d((n-1)!)} &= \prod_{p\mid n}\frac{w_p(n)+1}{w_p(n-1)+1}\\ &= \prod_{p^a\mid\mid n}\frac{w_p(n-1)+1+a}{w_p(n-1)+1}\\ &= \prod_{p^a\mid\mid n}\left(1 + \frac{a}{w_p(n-1)+1}\right)\\ &\le \exp\left(\sum_{p^a\mid\mid n}\frac{a}{w_p(n-1)+1}\right),\end{aligned}$$

where the final inequality comes from the fact that $\log(1+x) \leq x$ for all positive x. If $p \mid n$, then $\lfloor (n-1)/p \rfloor = n/p - 1$, so $w_p(n-1)+1 \geq \lfloor (n-1)/p \rfloor + 1 = n/p$ and thus

$$\frac{d(n!)}{d((n-1)!)} \leq \exp\left(\sum_{p^a \| n} \frac{a}{n/p}\right) = \exp\left(\frac{1}{n}\sum_{p^a \| n} ap\right) = \exp\left(\frac{S(n)}{n}\right).$$

Because $S(n)/n \leq 1$ and $e^x \leq 1+2x$ in the interval $(0,1]$, we have $d(n!)/d((n-1)!) \leq 1 + 2S(n)/n$ as desired.

We can prove the lower bound by noting that

$$\frac{d(n!)}{d((n-1)!} = \prod_{p^a \| n}\left(1 + \frac{a}{w_p(n-1)+1}\right) \geq 1 + \sum_{p^a \| n} \frac{a}{w_p(n-1)+1},$$

as all the terms in the sum on the right are contained in the expansion of the product. We can bound w_p above using a geometric series by removing the floors, so that $w_p(n-1)+1 < (n-1)\sum_{i=0}^{\infty} p^{-i} + 1 = 1 + (n-1)/(p-1) \leq 2n/p$. The last inequality holds for $2 \leq p \leq n$, with equality with $p = 2$ and $p = n$. Then we have

$$1 + \sum_{p^a \| n} \frac{a}{w_p(n-1)+1} = 1 + \sum_{p^a \| n} \frac{ap}{2n} = 1 + \frac{1}{2n}\sum_{p^a \| n} ap = 1 + S(n)/2n$$

as desired. ■

Theorem 2.4. *Let $P(n)$ be the largest prime divisor of n. Then we have*

$$\frac{d(n!)}{d((n-1)!)} = 1 + \frac{P(n)}{n} + O\left(n^{-1/2}\right).$$

Proof. Note that $S(n) \leq P(n)\Omega(n) \leq P(n)\log_2 n$, where the last inequality holds because 2 is the smallest prime and thus n can have at most $\log_2 n$ prime divisors. We split our proof into the cases where $P(n) \leq n^{1/2}$ and $P(n) > n^{1/2}$. First assume that $P(n) \leq n^{1/2}$. Let $p = P(n)$ and $q = P(n/p)$. If $q \leq n^{1/3}$, then $S(n) = p + S(n/p) \leq p + q\log_2 n = O\left(n^{1/2}\right)$. If $n^{1/3} < q \leq p$, then $S(n) = p + q + S(n/pq) \leq p + q + n/pq \leq 3p = O\left(n^{-1/2}\right)$. Thus in both cases $S(n) \leq n^{1/2}$, and since $P(n)/n$ is $O(n^{1/2})$ and we can bound $d(n!)/d((n-1)!)$ by multiples of $S(n)/n$, which we have proven to be $O(n^{-1/2})$, the theorem follows.

Now assume $p = P(n) > n^{1/2}$. Let m be such that $n = mp$, so that $w_p(n) = \lfloor n/p \rfloor = m$ and $w_p(n-1) = m-1$. Then

$$\begin{aligned}\frac{d(n!)}{d((n-1)!)} &= \prod_{p^a \| n}\left(1 + \frac{a}{w_p(n-1)+1}\right) \\ &= \left(\frac{m+1}{m}\right)\prod_{q^b \| m}\left(1 + \frac{b}{w_p(n-1)+1}\right).\end{aligned}$$

We now focus our attention towards the product. From the proof of Lemma 2.3, we know that

$$\begin{aligned}1 \le \prod_{q^b || m} \left(1 + \frac{b}{w_p(n-1)+1}\right) &\le \prod_{q^b || m} \left(1 + \frac{b}{n/q}\right) \\ &\le \exp\left(\sum_{q^b||m} \frac{bq}{n}\right) \\ &= \exp\left(\frac{S(m)}{n}\right) \\ &\le \exp\left(\frac{m}{n}\right) \\ &\le 1 + \frac{2m}{n} \\ &= 1 + O\left(\frac{1}{n^{1/2}}\right).\end{aligned}$$

Because $1/m = p/n = P(n)/n$ and

$$\frac{m+1}{m}\left(1 + O\left(\frac{1}{n^{1/2}}\right)\right) = 1 + \frac{1}{m} + O\left(\frac{1}{n^{1/2}}\right),$$

we have proven the theorem. ■

We will transition here to briefly discuss some results outlined in much more detail in [EGIP96] relating to the growth of $d(n!)$.

Definition 2.5. Let $K(n)$ be the smallest positive integer satisfying

$$d((n+K(n))!) \ge 2d(n!).$$

We will take the following theorem for granted in order to prove an interesting corollary that bounds $K(n)$.

Theorem 2.6. *Let $f(n)$ denote the smallest positive integer satisfying*

$$\sum_{i=1}^{f(n)} S(n+i) > n.$$

For each $\epsilon > 0$ there exist infinitely many integers satisfying

$$f(n) \ge (1/4 - \epsilon) \log n \log\log n \log\log\log\log n / (\log\log\log n)^3.$$

Additionally, if n is sufficiently large, then $f(n) < n^{\frac{4}{9}}$.

This theorem gives us the tools to show that $K(n)/\log n$ is unbounded.

Corollary 2.7. *We have*

$$K(n) > \frac{\log n \log\log n \log\log\log\log n}{9(\log\log\log n)^3}$$

for infinitely many positive integers n.

Proof. From Theorem 2.6, we know that there are infinitely many pairs n, K satisfying $K > \frac{1}{9}\log n \log\log n \log\log\log\log n/(\log\log\log n)^3$ and $\sum_{i=1}^{K} S(n+i) \le n/2$. The constant $\frac{1}{9}$ is not particularly important here as this statement is true for all values less than $\frac{1}{4}$. Because

$$\frac{d((n+K)!)}{d(n!)} = \prod_{i=1}^{K} \frac{d((n+i)!)}{d((n+i-1)!)} \le \exp\left(S(n+i)/(n+i)\right)$$

by techniques used to prove Lemma 2.3, we find that

$$\frac{d((n+K)!)}{d(n!)} \le \exp\left(\sum_{i=1}^{K} \frac{S(n+i)}{n+i}\right) < \exp\left(\frac{1}{n}\sum_{i=1}^{K} S(n+i)\right) \le e^{1/2} < 2.$$

Since for all $k > K(n)$ we must have $d((n+k)!)/d(n!) \ge 2$ (as $d((n+1)!) > d(n!)$ for all n), we know that $K(n) > K$ and thus we have proved our corollary. ■

We conclude this section by stating without proof the upper bound on $K(n)$ established in the paper.

Theorem 2.8. *For all sufficiently large integers n, $K(n) < n^{4/9}$.*

3. Asymptotics for $d(n!) - d((n-1)!)$

We now turn our attention to the differences between $d(n!)$ and $d((n-1)!)$.

Definition 3.1. We define

$$D(n) = d(n!) - d((n-1)!).$$

We can first use techniques developed earlier in the paper to bound $D(n)$.

Proposition 3.2. *We have the following:*

(1) $D(n) = d((n-1)!)\left(\frac{P(n)}{n} + O\left(n^{-1/2}\right)\right)$
(2) *There exist positive integers $d_0, d_1, \ldots$ such that*

$$\sum_{2\le n\le x} \log D(n) = \frac{x^2}{\log x}\sum_{k=0}^{K} \frac{d_k}{\log^k x} + O\left(\frac{x^2}{\log^{K+2} x}\right)$$

for any positive integer K.

Proof. For the first statement, we can write

$$D(n) = d((n-1)!)\left(\frac{d(n!)}{d((n-1)!)} - 1\right).$$

Applying Theorem 2.4, we have

$$D(n) = d((n-1)!)\left(\frac{P(n)}{n} + O\left(\frac{1}{n^{1/2}}\right)\right)$$

as desired. Taking logarithms, we find $\log D(n) = \log(d(n-1)!) + O(\log n)$ as $P(n)/n$ is $O(n)$. As a result, we sum the approximation from Theorem 1.2 to find that

$$\sum_{2\le n\le x} \log D(n) = \frac{x^2}{\log x}\sum_{k=0}^{K} \frac{d_k}{\log^k x} + O\left(\frac{x^2}{\log^{K+2} x}\right).$$

■

We finally investigate the occurrence of champs, a concept analogous to highly composite numbers.

Definition 3.3. We call a positive integer n a *champ* if $D(n) > D(m)$ for all positive integers $m < n$.

We can first prove the following result about champs.

Theorem 3.4. *For each prime p, both p and $2p$ are champs.*

Proof. For primes p, we have

$$D(p) = d(p!) - d((p-1)!) = d((p-1)!).$$

Because $d(m!)$ is an increasing function, we additionally have $d((p-1)!) \geq d(m!) > D(m)$ for all $m < p$. We can think about $D(m)$ as the number of divisors of $m!$ that do not divide $(m-1)!$. We know that there are at most as many of these as divisors of $(m-1)!$ as each divisor of this form is m multiplied by a divisor of $(m-1)!$. It follows that $D(m) \leq d(m!)/2$.

Now consider odd primes p. For these primes,

$$D(2p) = d((2p)!) - d((2p-1)!) \geq \frac{3}{2}d((2p-1)!) - d((2p-1)!) = \frac{1}{2}d((2p-1)!) \geq D(m)$$

for all $m < 2p$. Thus in this case $2p$ is a champ. We can verify that $4 = 2 \cdot 2$ is a champ as well, so that p and $2p$ are champs for all primes p. ∎

Categorizing all the champs proves very difficult, so we may ask instead how frequently champs occur. To do this, we introduce the concept of asymptotic density.

Definition 3.5. We define the asymptotic density of a set $S \subseteq \mathbb{N}$ to be

$$\lim_{n\to\infty} \frac{|\{x : x \in S, x \leq n\}|}{n}.$$

For instance, the set of multiples of n has asymptotic density $1/n$.

We hope to show that the set of champs has asymptotic density zero, which would imply that champs occur relatively infrequently. Note that the set of primes has asymptotic density zero as a consequence of the prime number theorem.

Theorem 3.6. *Assuming the Riemann Hypothesis, the set of champs has asymptotic density zero.*

The following lemma does not require assumption of the Riemann Hypothesis but forms an important piece of the proof of Theorem 3.6.

Lemma 3.7. *Recall that $P(n)$ is the largest prime divisor of n. If $P(n) \leq n/\log^3 n$ and there is a prime in the interval $(n - \frac{1}{3}\log^3 n, n]$, then n cannot be a champ.*

Proof. Assume on the contrary that n is a champ such that $P(n) \leq n/\log^3 n$. Letting $m = \left\lfloor n - \frac{1}{3}\log^3 n \right\rfloor$, we find that because n is a champ,

$$d((n-1)!) = d(m!) + D(m+1) + \ldots + D(n-1).$$

We can bound this above by $d(m!) + (n-1-m)D(n)$ as $D(n) > D(k)$ for $k < n$, which is at most $d(m!) + \frac{1}{3}D(n)\log^3 n$. By Theorem 2.4, we find that

$$d(n!) \leq \left(1 + \log^{-3} n + O\left(n^{1/2}\right)\right) d((n-1)!),$$

so that

$$D(n) \leq \left(\log^{-3} n + O\left(n^{1/2}\right)\right) d((n-1)!) \leq \frac{3}{2}(\log^{-3} n) d((n-1)!)$$

for sufficiently large n. Combining these two expressions, we find that

$$\begin{aligned} d((n-1)!) &< d(m!) + \frac{1}{3} D(n) \log^3 n \\ &= d(m!) + \frac{1}{3}\frac{3}{2}(\log^{-3} n)(\log^3 n) d((n-1)!) \\ &= d(m!) + \frac{1}{2} d((n-1)!), \end{aligned}$$

so that $d((n-1)!) < 2d(m!)$. If m were a prime, then $d(p!) = 2d((p-1)!)$, but since $d((n-1)!) < 2d((p-1)!)$, this would mean $d(p!) > d((n-1)!)$, a contradiction. Thus if n is a champ there are no primes in the interval $(m, n]$, so we have proven the lemma. ■

We are now ready to prove the theorem.

Proof of Theorem 3.6. By Lemma 3.7, the set of non-champs contains the intersection of the set of n such that $P(n) \leq n/\log^3 n$ and n such that there exists a prime in the interval $(n - \frac{1}{3}\log^3 n, n]$. Selberg proved in [Sel43] that the second set has asymptotic density 1 assuming the Riemann Hypothesis. We will show that the first set has asymptotic density 1 regardless of the validity of the Riemann Hypothesis. We can do this by showing that its complement, the set of n such that $P(n) > n/\log^3 n$, has asymptotic density zero. Consider this set for $n \leq x$. We have $P(n) \cdot \log^3 n > n$, and $n = P(n) \cdot q$. This means that for all $q < \log^3 n$, our condition is satisfied. However, it may be possible that $P(n) \cdot \log^3 n > x$, which occurs when $n > x/\log^3 x$. We can split this sum when this happens to find that the number of elements we are looking for is equal to

$$\begin{aligned} \sum_{\substack{p \leq x/\log^3 x \\ p \text{ prime}}} (\log^3 p - 1) + \sum_{\substack{p > x/\log^3 x \\ p \text{ prime}}} \left\lfloor \frac{x}{p} \right\rfloor &= \sum_{\substack{p \leq x/\log^3 x \\ p \text{ prime}}} (\log^3 p - 1) + x \sum_{\substack{p > x/\log^3 x \\ p \text{ prime}}} \frac{1}{p} + o(x) \\ &\leq O\left(\frac{x}{\log^3 x \log\left(\frac{x}{\log^3 x}\right)} \log^3 x\right) + x \sum_{\substack{p > x/\log^3 x \\ p \text{ prime}}} \frac{1}{p} + o(x) \\ &= x \sum_{\substack{p > x/\log^3 x \\ p \text{ prime}}} \frac{1}{p} + o(x) \\ &= O\left(x(\log\log x - \log\log(x/\log^3 x))\right) + o(x) \\ &= O\left(x \log\left(\frac{\log x}{\log(x/\log^3 x)}\right)\right) + o(x). \end{aligned}$$

Taking the limit as $x \to \infty$, we find that $\log x/(\log(x/\log^3 x))$ tends to 1, which means that the logarithm of this expression tends to zero and thus the first term is $o(x)$. Becauuse the entire set is $o(x)$, it follows that it has asymptotic density zero and thus its complement has asymptotic density 1. This completes the proof of the theorem. ■

4. Conclusion

Erdös, Pomerance, Graham, and Ivíc conclude the paper by stating various conjectures, most interestingly that the set of n such that $D(n+1) > D(n)$ has asymptotic density 1/2. Additionally, they continue to seek an unconditional proof that the set of champs has asymptotic density zero.

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