Exploring the Riemann zeta function

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Abstract

An important function in studying primes is the *Riemann zeta function*. The locations of its zeros are deeply connected to approximating primes. Here, we'll explore why there's a connection, what the *Riemann hypothesis* can tell us about this, and some other consequences of the Riemann hypothesis.

1 Introduction

In this paper, we'll talk about Riemann's Explicit Formula. There's a big connection between the distribution of primes and the Riemann zeta function.

Definition 1.1 The *Riemann zeta function* $\zeta(s)$ is defined to be

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Now, the Riemann zeta function is undefined for $s \leq 1$: It diverges to ∞. However, there's a way of extending the zeta function to be defined on all numbers except 1 (it goes to ∞ there). Actually, if we allow for complex exponents, the zeta function is defined on the complex numbers whose real parts are greater than 1, and the extension is defined on all *complex* numbers except 1.

What we're interested in is the zeros of this (extended) function: The complex numbers s such that $\zeta(s) = 0$. There's the *trivial zeros*, which are the negative even integers, $-2, -4, -6, \cdots$. There have been other, non-trivial zeros found on the line of real part $\frac{1}{2}$, that is, their real part is $\frac{1}{2}$, but none found elsewhere. However, it hasn't been proven or disproved that there are none elsewhere.

Hypothesis 1.2 The *Riemann hypothesis* is that all nontrivial zeros of the zeta function have real part $\frac{1}{2}$.

There's a nice relationship between the zeros of $\zeta(s)$ and the *prime counting function* $\pi(x)$, which is the function that tells you the number of primes $\leq x$ (this has nothing to do with the number π). It involves taking *Riemann residues* of complex numbers. I won't really get into the details of them here, but the point is this: Riemann residues turn complex numbers into functions on $\mathbb{R}$. When you subtract the Riemann residues of the zeta zeros from that of 1, it turns out $\pi(x)$ is what you get (see [2])! (Note that this involves limits since there are infinitely many zeros.)

Theorem 1.3 The Riemann residue for 1, called $\text{li}(x)$, minus the Riemann residues for the zeros of the zeta function, is exactly the function $\pi(x)$.

Now, $\text{li}(x)$ is a relatively good approximation for $\pi(x)$, the residues of the zeta zeros being the error terms. If the Riemann hypothesis is true, meaning all the non-trivial zeros lie on the line of real part $\frac{1}{2}$ (called the *critical line*), then it turns out this makes their Riemann residues very small, and $\text{li}(x)$ approximates $\pi(x)$ with very little error! However, just one single nontrivial zero even a little off would have a devastating effect on this approximation: It would create a large amount of error.

2 What we know

What do we know about the zeros of the zeta function? Well, we know that the only zeros with nonpositive real part are the trivial ones. We also know that there's no zeros with real part ≥ 1. Thus all nontrivial zeros lie in what we call the *critical strip*:

Definition 2.1 The *critical strip* is the subset of the complex numbers containing the numbers of real part strictly between 0 and 1.

The real part of a complex number is related to how it grows. If we take a complex number $a + bi$, it turns out that its Riemann converter can be approximated by

$$\frac{x^a}{\log(x)\sqrt{a^2 + b^2}} \cdot \cos(b \log(x) - \arctan(b/a)).$$

This seems like a very complicated expression, but the main thing is the x^a: It represents the growth of the function. You could say that the Riemann converter grows like x^a. If the zeros all lie on the critical line, then $a = \frac{1}{2}$, so they all grow like $\sqrt{x}$. For reasons we won't get into here, this means the whole error term grows like $\sqrt{x}\log(x)$ (the trivial zero terms just go to 0), and in fact something it shows is that

$$|\pi(x) - \mathrm{li}(x)| \leq \frac{\sqrt{x}}{8\pi} \log(x).$$

(They're actually equivalent!) But, if we had one single zero whose real part was just a bit above 1/2, this approximation would be completely shattered! Let its real part be $a > \frac{1}{2}$. Then $\sqrt[a]{x}$, even times some really small constant, would eventually get way bigger than $\sqrt{x}$, because it has a higher growth rate. So our approximation would be very wrong if this happened!

Even so, there would still be hope. Let's take another look at what we know. The part about nontrivial zeros not having real part ≤ 0 isn't that important. Those terms would already be dominated by the nontrivial zeros on the critical line that we know. What's important is that there's none with real part ≥ 1. This tells us that the growth rate of the zeta zero terms is at most x^1, or

Theorem 2.2 The *Prime Number Theorem* states that

$$\pi(x) \sim \mathrm{li}(x).$$

Actually, you can further approximate $\mathrm{li}(x)$ as $\frac{x}{\log(x)}$, so the usual form it's written in is

$$\pi(x) \sim \frac{x}{\log(x)}.$$

And even if the Riemann hypotheses is false, or at least we can't prove it true, perhaps we can still bound it further: We could show no zero has real part greater than a for some $a \geq \frac{1}{2}$. Then, the Riemann converters of the zeta zeros have growth rate at most $\sqrt[a]{x}$. The smaller we make a, the better an approximation we show $\mathrm{li}(x)$ to be. However, the cases with $a > \frac{1}{2}$ aren't quite as good as the Riemann hypothesis.

3 A generalization of the Riemann zeta function

I've mentioned that the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\zeta(s)$ only converges if $\Re(s) > 1$, where $\Re(s)$ denotes the real part of s. I've also mentioned that there's actually a way of extending this function defined on all of $\mathbb{C}$ except 1. Now, you might be thinking, "aren't there a bunch of different extensions you could use, so why are we using one specific one?" Well, yes, you could extend the zeta function any which way you like, giving any new input any value you want to. But there's one particular extension that has something special about it: It's complex differentiable everywhere.

What do I mean by this? Well, you can take derivatives of complex functions like with real-valued functions. I won't get into that here, but the point is this: The original zeta function is complex differentiable everywhere it's defined, so we want the extension to do that too. And it turns out there's only one function that does that, called the *analytic continuation* of the zeta function. (See [1])

You might wonder if there's a formula for this analytic continuation. Indeed there is!

Proposition 3.1 First, let $\zeta_a(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$. Then $\zeta(s) = \left(1 - \frac{1}{2^{s-1}}\right)^{-1} \zeta_a(s)$ is defined for all s with real part > 0 and unequal to $1 + \frac{2}{\log(2)}\pi ik$ for some integer k. But with these, except for 1, you can take limits and you will get well-defined values. Now, the rest of the values can be obtained from these using the following formula:

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{s\pi}{2} \Gamma(1-s)\zeta(1-s).$$

This extension is complex differentiable everywhere.

The Riemann hypothesis is about this analytic continuation of the zeta function, since where the original zeta function is defined, there's no zeros as we've seen.

Now, the Riemann hypothesis is not only useful for understanding primes. There are so many results that have been proven assuming the Riemann hypothesis, and some that have been shown to be equivalent to it! So many fields of study in quantum physics, cryptography, and others assume the Riemann hypothesis, so there's a lot hinging on the truth of this one single statement. And if it's proven false it might shatter all of this!

There's also some things that have been proven true assuming the *Generalized Riemann Hypothesis*. See, the function $\zeta(s)$ can be generalized. First, we need to define some things:

Definition 3.2 A *Dirichlet character* is a function $\chi : \mathbb{N} \to \mathbb{C}$ with the following properties:

- $\chi(mn) = \chi(m)\chi(n)$ for all m, n (that is, χ is *completely multiplicative*)
- There exists a positive integer k such that $\chi(n+k) = \chi(n)$ for all n and $\chi(n) = 0$ whenever $\gcd(n, k) > 1$. This is called the *modulus* of χ.

Definition 3.3 The *Dirichlet L-function*, $L(\chi, s)$ for character χ and variable s is defined as

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Note that since the function $\chi(n) = 1$ is a character, and this χ gives $L(\chi, s) = \zeta(s)$, $\zeta(s)$ is just a special case of these L-functions. In fact, these converge when $\Re(s) > 1$, and they have analytic continuations just like $\zeta(s)$! And just like the zeta function, they do have a few points with undefined values, but those don't matter that much here. What matters is how their zeros behave:

Hypothesis 3.4 The *Generalized Riemann Hypothesis* states that for every Dirichlet character χ and every complex number s with $L(\chi, s) = 0$, if s is not a negative real number, then $\Re(s) = \frac{1}{2}$.

Let's look at some results that have been proven assuming the Ceneralized Riemann hypothesis. First:

Theorem 3.5 *Dirichlet's Theorem* states that for any coprime positive integers a and d, the set $\{a, a+d, a+2d, a+3d, \ldots\}$ contains infinitely many prime numbers.

You can strengthen this using the Generalized Riemann Hypothesis:

Theorem 3.6 If the Generalized Riemann Hypothesis is true, then

$$\pi(x, a, d) = \frac{\mathrm{li}(x)}{\phi(d)} + O(x^{\frac{1}{2}+\epsilon})$$

for any $\epsilon > 0$, where $\pi(x, a, d)$ denotes the number of primes in our set $\{a, a + d, \dots\}$ that are $\leq x$.

(See [3])

We won't get into here what the O means, but basically the O term is just giving a bound on the amount of error.

There's actually a weaker version of this that can be proven unconditionally (i.e. not assuming anything):

Theorem 3.7 Take a, d as before but additionally, $d \geq 3$. Then

$$\left|\pi(x, a, d) - \frac{\mathrm{li}(x)}{\phi(d)}\right| < \frac{Ax}{(\log x)^2}$$

for all $x \geq B$, where $A = \frac{1}{840}$ if $3 \leq d \leq 10^4$, $A = \frac{1}{160}$ if $d > 10^4$, $B = 8 \cdot 10^9$ if $3 \leq d \leq 10^5$, and $B = e^{0.03\sqrt{d}(\log d)^3}$ if $d > 10^5$.

Another thing implied by the Generalized Riemann Hypothesis is the Goldbach Conjecture:

Conjecture 3.8 The *Goldbach Conjecture* states that all even numbers greater than 2 can be expressed as the sum of two prime numbers.

Now, there's actually useful things that can be proven true assuming the Generalized Riemann Hypothesis, or something weaker, is *false*. One example is what's called the *Heath-Brown Theorem*:

Theorem 3.9 The *Heath-Brown Theorem* states that either *Siegel zeros* don't exist or the Twin Prime Conjecture is true (!)

(See [5])

Now, a Siegel zero is a special kind of (potential) counterexample to the Generalized Riemann hypothesis. We won't get into the precise definition of it here, but basically it's a real number close to 1 that is a zero of some L-function.

This is the Twin prime conjecture for those who don't know about it:

Conjecture 3.10 The *Twin Prime Conjecture* states that there are infinitely many pairs of primes of the form $(p, p + 2)$. These primes are called *Twin Primes*.

So if there exists just one Siegel zero, that automatically proves the Twin Prime Conjecture! Or if there's only finitely many pairs of twin primes, that would show there are no Siegel zeros.

4 Other values of the zeta function

You might wonder what other values of $\zeta(s)$ besides the zeros do we know. Well, let's take a look at the integer values. For starters, we've seen that $\zeta(1)$ is undefined; that's the point at which it diverges to ∞. We also know the values $\zeta(n)$ for even positive integers n. You may have seen that $\zeta(2) = \frac{\pi^2}{6}$. Perhaps you've also seen that $\zeta(4) = \frac{\pi^4}{90}$, or that $\zeta(6) = \frac{\pi^6}{945}$. Well, there's actually a formula for all of these:

Theorem 4.1 If n is an even positive integer, then

$$\zeta(n) = (-1)^{\frac{n}{2}+1} \frac{(2\pi)^n B_n}{2(n!)},$$

where B_n are the *Bernoulli numbers*.

(See [4])

We also know things about the nonpositive integers:

Theorem 4.2 If n is a nonpositive integer, then

$$\zeta(n) = (-1)^n \frac{B_{-n+1}}{-n+1}.$$

Remember, this is using the analytic continuation of the zeta function. They all diverge to ∞ if we use the original definition.

We know that the negative even integers always give 0. The first few negative odd values are $\zeta(-1) = -\frac{1}{12}$, $\zeta(-3) = \frac{1}{120}$, $\zeta(-5) = -\frac{1}{252}$, etc. Also, $\zeta(0) = -\frac{1}{2}$.

This just leaves the odd positive integers other than 1. What are those values? Well, we actually don't have explicit formulas for them. They're just new constants. For example, $\zeta(3)$ is called Apéry's constant. It's been shown that $\zeta(3)$ is irrational, but it's still unknown whether it's transendental. It's also been shown that infinitely many of these are irrational, and that at least one of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, or $\zeta(11)$ is irrational, but we're still not sure which. However, in all likelihood, all of those are transcendental.

As you can see, there are a lot of mysteries around the Riemann zeta function. There's also a lot of nice stuff we can prove assuming the Riemann hypothesis is true. For now, all we can do is keep learning new things and keep exploring!

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