APERY'S THEOREM

ARJUN AGARWAL

1. Introduction

Apery's theorem is a well known result in analytic number theory that states that $\zeta(3)$, the Reimann zeta function evaluated at 3 is irrational. Mathematically

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} \neq pq$$

where $p, q \in \mathbb{Z}$. $\zeta(3)$ is also known as Apery's constant. The theorem is named after the mathematician Roger Apery, who first provided an outline of the proof for this theorem. The special values of the Riemann zeta function at even integers can be shown to be irrational. It still remains an open problem whether the Reimann zeta function is rational or not (they are conjectured to be irrational) for odd integers greater than 3. This expository paper outlines the proof provided in Alfred van der Poorten's report titled *A Proof that Euler Missed ... Apery's proof of the Irrationality of* $\zeta(3)$. The bulk of the proof was provided by Cohen following Apery's outline.

2. Apery's Proof

Theorem 2.1. A constant β is irrational if there exists a $\delta > 0$ and a sequence $\{\frac{p_n}{q_n}\}$ of rational numbers such that $\frac{p_n}{q_n} \neq \beta$ and

$$|\beta - \frac{p_n}{q_n}| < \frac{1}{q_n^{1+\delta}} \quad n = 1, 2, \ldots$$

Apery's proof seeks to use manipulation of sequences and this theorem to prove $\zeta(3)$ is irrational.

Definition 2.2. Define a recursion as follows

$$n^3 u_n + (n-1)^3 u_{n-1} = (34n^3 - 51n^2 + 27n - 5)u_{n-1}, \ n \geqslant 2$$

Also, define $\{b_n\}$ such that $b_0 = 1, b_1 = 5$ and $b_n = u_n$ for $n \geqslant 2$. Additionally, let $\{a_n\}$ be defined as $a_0 = 0, a_1 = 6$ and $a_n = u_n$ for $n \geqslant 2$. Then $b_n \in \mathbb{Z}$ and $a_n \in \mathbb{Q}$ with the denominator dividing $2[1, 2, \ldots, n]^3$ where $[1, 2, \ldots, n]$ is the least common multiple of $1, 2, \ldots, n$.

Now, let

$$c_{n,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}}$$

Clearly, $c_{n,k} \to \zeta(3)$ as $n \to \infty$. However, the irrationality criteria cannot be applied on $c_{n,k}$ as the sequence does not converge fast enough. The following process can be used to accelerate the convergence Consider two triangular arrays defined for $k \leqslant n$ with entries $d^{(0)}_{n,k} = c_{n,k}\binom{n+k}{k}$ and $\binom{n+k}{k}$. Given any diagonal, the quotient of the corresponding elements of these arrays tends to $\zeta(3)$. Now consider the following series of transformations

$$\begin{array}{c|c}
d^{(0)}_{n,k} \to d^{(0)}_{n,n-k} = d^{(1)}_{n,k} & \binom{n+k}{k} \to \binom{2n-k}{n} \\[2ex]
d^{(1)}_{n,k} \to \binom{n}{k} d^{(1)}_{n,k} = d^{(2)}_{n,k} & \binom{2n-k}{n} \to \binom{n}{k}\binom{2n-k}{n} \\[2ex]
d^{(2)}_{n,k} \to \sum_{k'=0}^{k} \binom{k}{k'} d^{(2)}_{n,k} = d^{(3)}_{n,k} & \binom{n}{k}\binom{2n-k}{n} \to \sum_{k_1=0}^{k} \binom{k}{k_1}\binom{n}{k_1}\binom{2n-k_1}{n} \\[2ex]
d^{(3)}_{n,k} \to \binom{n}{k} d^{(3)}_{n,k} = d^{(4)}_{n,k} & \sum_{k_1=0}^{k} \binom{k}{k_1}\binom{n}{k_1}\binom{2n-k_1}{n} \to \sum_{k_1=0}^{k} \binom{k}{k_1}\binom{n}{k_1}\binom{n}{k}\binom{2n-k_1}{n} \\[2ex]
d^{(4)}_{n,k} \to \sigma_{k'=0}^{k} \binom{k}{k'} d^{(4)}_{n,k} = d^{(5)}_{n,k} & \sum_{k_2=0}^{k} \sum_{k_1=0}^{k_2} \binom{k}{k_2}\binom{k_2}{k_1}\binom{n}{k_1}\binom{n}{k_2}\binom{2n-k_1}{n}
\end{array}$$

The series of transformations above still retain the property that the quotient of the corresponding diagonal elements converges to $\zeta(3)$. Taking the main diagonals of the two arrays, we get the definitions of $\{a_n\}$ and $\{b_n\}$ from Definition 2.2. In addition $\dfrac{a_n}{b_n} \to \zeta(3)$. Now let $P(n-1) = 34n^3 - 51n^2 + 27n - 5$. Based on definitions of a_n, b_n

$$n^3 a_n - P(n-1)a_{n-1} + (n-1)^3 a_{n-1} = 0$$
$$n^3 b_n - P(n-1)b_{n-1} + (n-1)^3 b_{n-1} = 0$$

Eliminating $P(n-1)$,

$$n^3(a_n b_{n-1} - a_{n-1}b_n) = (n-1)^3(a_{n-1}b_{n-2} - a_{n-2}b_{n-1}).$$

Manipulating with $a_1 b_0 - a_0 b_1 = 6$

$$a_n b_{n-1} - a_{n-1} b_n = \frac{6}{n^3}.$$

This gives

$$\frac{a_n}{b_n} - \frac{a_{n-1}}{b_{n-1}} = \frac{6}{n^3 b_{n-1} b_n}$$

Also by definition of $a_0 = 0$, we have $\zeta(3) - \dfrac{a_0}{b_0} = \zeta(3)$. Combining with the above equation we have

$$|\zeta(3) - \frac{a_n}{b_n}| = \sum_{k=n+1}^{\infty} \frac{6}{k^3 b_k b_{k-1}}$$

Therefore, $\zeta(3) - \dfrac{a_n}{b_n} = O(b_n^{-2})$. Based on the recurrence for b_n we have

$$b_n - (34 - \frac{51}{n} + \frac{27}{n^2} - \frac{5}{n^3})b_{n-1} + (1 - \frac{3}{n} + \frac{3}{n^2} - \frac{1}{n^3})b_{n-2} = 0$$

Asymptotically, since $x^2 - 34x + 1$ has zeros $(1 \pm \sqrt{2})^4$, we have $b_n = O(\alpha^n)$ where $\alpha = (1 + \sqrt{2})^4$. Furthermore since a_n is not an integer, define

$$p_n = 2[1, 2, \ldots, n]^3 a_n, q_n = 2[1, 2, \ldots, n]^3 b_n$$

Then $p_n, q_n \in \mathbb{Z}$ and $q_n = O(\alpha^n e^{3n})$ and

$$\zeta(3) - \frac{p_n}{q_n} = O(\alpha^{-2n}) = O(q_n^{-(1+\delta)})$$

with $\delta = \dfrac{\log(\alpha) - 3}{\log(\alpha) + 3} \approx 0.0805 \cdots > 0$. Therefore by irrationality criteria $\zeta(3)$ is irrational.

Email address, A. Agarwal: `arjunagarwal010@gmail.com`

Euler Circle, Palo Alto, CA, 94306