Apery's Theorem

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1 Introduction

The Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is a well-studied object in number theory owing to its deep connection to prime numbers. However, evaluating this function at specific points is a difficult task. We know the values of the zeta function at negative numbers (0 for even negative numbers and a rational number otherwise), as well as the values at positive even numbers (a rational number times a power of π). However, computation of the zeta function at *any* odd integer > 1 has eluded us. The main topic of this expository paper–Apery's theorem–is a step toward the computation of these values, proving that $\zeta(3)$ is irrational.

2 Proving Apery's theorem

Definition 2.1 (The Riemann zeta function)**.** The *Riemann zeta function* (or just the *zeta function*) is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This formula can be extended via analytic continuation to all values of s except for 1, though this is beyond the scope of this paper.

Theorem 2.2 (Apery's Theorem)**.** $\zeta(3)$ *is irrational.*

Our method of attack will be the following well-known irrationality criterion.

Theorem 2.3. *For a real number α, if there exist integer sequences $(p), (q)$ such that*

$$\alpha \neq \frac{p_n}{q_n}$$

for all n and

$$\lim_{n \to \infty} |\alpha q_n - p_n| = 0,$$

α is irrational.

Essentially, rational numbers can approximate irrational numbers much better than they can approximate rational numbers.

Let us start by rewriting $\zeta(3)$:

Theorem 2.4. *We have*

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}.$$

We will require use of the following lemma:

Lemma 2.5. *For a sequence $a_1, a_2, \ldots, a_k$, we have*

$$\sum_{i=1}^{k} \frac{\prod_{1\le j\le i-1} a_j}{\prod_{1\le j\le i}(x+a_j)} = \frac{1}{x} - \frac{\prod_{1\le i\le k} a_i}{x\prod_{1\le i\le k}(x+a_i)}.$$

Proof. We will show this via induction. When $k = 0$, both sides are equal to 0. Then, if we assume the result for $k-1 \ge 0$, for k, it suffices to show:

$$\frac{\prod_{1\le i\le k-1} a_i}{\prod_{1\le i\le k}(x+a_i)} = \frac{\prod_{1\le i\le k-1} a_i}{x\prod_{1\le i\le k-1}(x+a_i)} - \frac{\prod_{1\le i\le k} a_i}{x\prod_{1\le i\le k}(x+a_i)}.$$

We have

$$\begin{aligned}
\frac{\prod_{1\le i\le k-1} a_i}{x\prod_{1\le i\le k-1}(x+a_i)} - \frac{\prod_{1\le i\le k} a_i}{x\prod_{1\le i\le k}(x+a_i)} &= \frac{\prod_{1\le i\le k-1} a_i}{x\prod_{1\le i\le k-1}(x+a_i)}\left(1 - \frac{a_i}{x+a_k}\right)\\
&= \frac{\prod_{1\le i\le k-1} a_i}{x\prod_{1\le i\le k-1}(x+a_i)} \cdot \frac{x}{x+a_k}\\
&= \frac{\prod_{1\le i\le k-1} a_i}{\prod_{1\le i\le k}(x+a_i)}
\end{aligned}$$

as desired. □

Now, we may prove Theorem 2.4:

Proof. We will use the setup of Lemma 2.5. Let $x = n^2$ and let $a_k = -k^2$. Then, we have

$$\begin{aligned}
\sum_{i=1}^{n} \frac{\prod_{1\le j\le i-1} a_j}{\prod_{1\le j\le i}(x+a_j)} &= \sum_{i=1}^{n} \frac{(-1)^{i-1}(i-1)!^2}{\prod_{1\le j\le i}(n^2-i^2)}\\
&= \frac{1}{n^2} - \frac{(-1)^{n-1}(n-1)!^2}{n^2\prod_{1\le i\le n}(n^2-i^2)}\\
&= \frac{1}{n^2} - \frac{(-1)^{n-1}(n-1)!^2}{n^2(1)(2)\cdots(n-1)(n+1)(n+2)\cdots(2n-1)}\\
&= \frac{1}{n^2} - \frac{2(-1)^{n-1}n!}{n^2(n+1)(n+2)\cdots(2n-1)(2n)}\\
&= \frac{1}{n^2} - \frac{2(-1)^{n-1}}{n^2\binom{2n}{n}}.
\end{aligned}$$

Next, we will massage $\sum_{i=1}^{n} \frac{(-1)^{i-1}(i-1)!^2}{\prod_{1\le j\le i}(n^2-i^2)}$ into a more useful form. In particular, consider

$$c_{n,k} = \frac{k!^2(n-k)!}{2k^3(n+k)!}.$$

Then, we have

$$
\begin{aligned}
(-1)^k n\left(c_{n,k}-c_{n-1,k}\right) &= \frac{(-1)^k n}{2}\left(\frac{k!^2(n-k)!}{k^3(n+k)!}-\frac{k!^2(n-k-1)!}{k^3(n+k-1)!}\right) \\
&= \frac{(-1)^k n}{2}\cdot\frac{k!^2(n-k-1)!}{k^3(n+k-1)!}\left(\frac{n-k}{n+k}-1\right) \\
&= \frac{(-1)^{k-1} n}{2}\cdot\frac{k!^2(n-k-1)!}{k^3(n+k-1)!}\left(1-\frac{n-k}{n+k}\right) \\
&= \frac{(-1)^{k-1} n}{2}\cdot\frac{k!^2(n-k-1)!}{k^3(n+k-1)!}\cdot\frac{2k}{n+k} \\
&= (-1)^{k-1} n\cdot\frac{(k-1)!^2}{(n-k)(n-k+1)\cdots(n+k)} \\
&= \frac{(-1)^{k-1}(k-1)!^2}{\prod_{i=1}^k\left(n^2-i^2\right)}
\end{aligned}
$$

Now, note that

$$
\sum_{n=1}^m\sum_{k=1}^{n-1}\frac{(-1)^k}{2}\left(\frac{k!^2(n-k)!}{k^3(n+k)!}-\frac{k!^2(n-k-1)!}{k^3(n+k-1)!}\right)=\sum_{n=1}^m\frac{1}{n^3}-2\sum_{n=1}^m\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.
$$

Summing the LHS a different way gives

$$
\begin{aligned}
\sum_{n=1}^m\sum_{k=1}^{n-1}(-1)^k(c_{n,k}-c_{n-1,k}) &= \sum_{k=1}^{m-1}(-1)^k\sum_{n=k+1}^m(c_{n,k}-c_{n-1,k}) \\
&= \sum_{k=1}^{m-1}(-1)^k(c_{m,k}-c_{k,k}) \\
&= \sum_{k=1}^{m}(-1)^k(c_{m,k}-c_{k,k}) \\
&= \sum_{k=1}^m\frac{(-1)^k k!^2(m-k)!}{2k^3(m+k)!}+\sum_{k=1}^m\frac{(-1)^{k-1}k!^2}{2k^3(2k)!}.
\end{aligned}
$$

Note that

$$
\lim_{m\to\infty}\sum_{k=1}^m\frac{(-1)^k k!^2(m-k)!}{2k^3(m+k)!}=0
$$

since the terms are $O(m^{-2})$ and we are only summing up m of them. Then,

$$
\sum_{n=1}^\infty\frac{1}{n^3}-2\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}=\frac{1}{2}\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.
$$

so

$$
\sum_{n=1}^\infty\frac{1}{n^3}=\frac{5}{2}\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}
$$

as desired. □

Next, consider the following sequence:

Definition 2.6. Let

$$
t_{n,k}=\sum_{m=1}^n\frac{1}{m^3}+\sum_{m=1}^k\frac{(-1)^{m-1}}{2m^3\binom{n}{m}\binom{n+m}{m}}=\sum_{m=1}^n\frac{1}{m^3}+\sum_{m=1}^k c_{n,m}.
$$

As $n \to \infty$, the second term goes to 0 uniformly for all k. Hence, some subsequence of this triangle could be used as convergents for $\zeta(3)$. However, this convergence is not fast enough. Instead, consider the following sequences.

Definition 2.7. Define the sequence a of rational numbers as follows:

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 6 \\ a_n &= \frac{(34n^3 - 51n^2 + 27n - 5)a_{n-1} - (n-1)^3 a_{n-2}}{n^3}. \end{aligned}$$

It turns out that

$$a_n = \sum_{k=0}^{n} t_{n,k} \binom{n}{k}^2 \binom{n+k}{k}^2,$$

though proving this is beyond the scope of this paper.

Definition 2.8. Define the sequence b of rational numbers as follows:

$$\begin{aligned} b_0 &= 1 \\ b_1 &= 5 \\ b_n &= \frac{(34n^3 - 51n^2 + 27n - 5)b_{n-1} - (n-1)^3 b_{n-2}}{n^3}. \end{aligned}$$

It turns out that

$$b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2,$$

though proving this is beyond the scope of this paper.

Lemma 2.9. *For all $n \geq 0$,*

$$2\text{lcm}(1, 2, \ldots, n)^3 a_n \in \mathbb{Z}.$$

Proof. Recall that

$$a_n = \sum_{k=0}^{n} t_{n,k} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \sum_{m=1}^{n} \frac{1}{m^3} \binom{n}{k}^2 \binom{n+k}{k}^2 + \sum_{k=0}^{n} \sum_{m=1}^{k} c_{n,m} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

The first term on the RHS is clearly in $\frac{\mathbb{Z}}{2\text{lcm}(1,2,\ldots,n)^3}$, so it suffices to show that the second term satisfies a similar property. We have

$$\sum_{k=0}^{n} \sum_{m=1}^{k} c_{n,m} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \sum_{m=1}^{k} \frac{(-1)^{m-1} \binom{n}{k}^2 \binom{n+k}{k}^2}{2m^3 \binom{n}{m} \binom{n+m}{m}}.$$

Consider the number of times that a prime p divides each term (denoted by the function ν_p). We will show that this value is always nonnegative. First, we will show that

$$\nu_p\left(\binom{n}{m}\right) \leq \nu_p\left(\text{lcm}(1, 2, \ldots, n)\right) - \nu_p(m).$$

To do this, compare the products

$$(m)(m+1)\cdots(n),$$

and

$$(1)(2)\cdots(m).$$

Say that k has the maximum value of $v_p(k)$ among all values between m and n. Then, the other values between m and n have sum of value of v_p at most $v_p\left(\operatorname{lcm}(1,2,\ldots,n)\right)$, which means that

$$v_p\left(m\binom{n}{m}\right) \le v_p\left(\operatorname{lcm}(1,2,\ldots,n)\right),$$

which gives the desired result.

Now, we show that

$$2\operatorname{lcm}(1,2,\ldots,n)\frac{(-1)^{m-1}\binom{n+k}{k}}{2m^3\binom{n}{m}\binom{n+m}{m}}$$

is an integer by counting the factors of p:

$$\begin{aligned}
v_p\left(2\operatorname{lcm}(1,2,\ldots,n)\frac{(-1)^{m-1}\binom{n+k}{k}}{2m^3\binom{n}{m}\binom{n+m}{m}}\right) &= 3\left\lfloor\frac{\log n}{\log p}\right\rfloor - 3v_p(m) + v_p\left(\frac{\binom{n+k}{k}}{\binom{n}{m}\binom{n+m}{m}}\right)\\
&= 3\left\lfloor\frac{\log n}{\log p}\right\rfloor - 3v_p(m) + v_p\left(\frac{\binom{n+k}{k-m}}{\binom{n}{m}\binom{k}{m}}\right)\\
&\ge 3\left\lfloor\frac{\log n}{\log p}\right\rfloor - 3v_p(m) + v_p\left(\frac{1}{\binom{n}{m}\binom{k}{m}}\right)\\
&\ge 3\left\lfloor\frac{\log n}{\log p}\right\rfloor - 3v_p(m) - \left\lfloor\frac{\log n}{\log p}\right\rfloor + v_p(m) - \left\lfloor\frac{\log k}{\log p}\right\rfloor + v_p(m)\\
&= \left\lfloor\frac{\log n}{\log p}\right\rfloor - \left\lfloor\frac{\log k}{\log p}\right\rfloor + \left\lfloor\frac{\log n}{\log p}\right\rfloor - v_p(m)\\
&\ge 0.
\end{aligned}$$

□

Lemma 2.10. *For all $n \ge 0$,*

$$b_n \in \mathbb{Z}.$$

This follows trivially from the combinatorial sum for b_n. Now, we may show the following:

Lemma 2.11. *We have*

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \zeta(3).$$

Proof. Note that $\frac{a_n}{b_n}$ is a weighted average of $c_{n,0}, c_{n,1}, \ldots, c_{n,n}$. Thus, since $c_{n,k}$ converges uniformly in k to $\zeta(3)$ as $n \to \infty$, this weighted average also converges to $\zeta(3)$. □

We now present the proof of Apery's theorem!

Proof of Theorem 2.2. We will show that

$$\left|\zeta(3) - \frac{a_n}{b_n}\right| = O\left(\left(2\operatorname{lcm}(1,2,\ldots,n)^3 b_n\right)^{-\alpha}\right)$$

for $\alpha > 1$, so we will be done by Theorem 2.3.

Let $P = 34x^3 - 51x^2 + 27x - 5$, so that

$$a_n = \frac{P(n)a_{n-1} - (n-1)^3 a_{n-2}}{n^3}$$

and analogously for b. Then,

$$\begin{aligned}
a_n b_{n-1} - a_{n-1}b_n &= \frac{\left(P(n)a_{n-1}b_{n-1} - (n-1)^3 a_{n-2}b_{n-1}\right) - \left(P(n)b_{n-1}a_{n-1} - (n-1)^3 b_{n-2}a_{n-1}\right)}{n^3}\\
&= \frac{(n-1)^3\left(a_{n-1}b_{n-2} - b_{n-1}a_{n-2}\right)}{n^3}\\
&= \frac{a_1 b_0 - b_1 a_0}{n^3}\\
&= \frac{6}{n^3}.
\end{aligned}$$

Now,

$$\begin{aligned}\zeta(3) - \frac{a_n}{b_n} &= \sum_{k=n+1}^{\infty} \frac{a_k}{b_k} - \frac{a_{k-1}}{b_{k-1}} \\ &= \sum_{k=n+1}^{\infty} \frac{a_k b_{k-1} - a_{k-1} b_k}{b_k b_{k-1}} \\ &= \sum_{k=n+1}^{\infty} \frac{6}{k^3 b_k b_{k-1}} \\ &= O(b_n^{-2}).\end{aligned}$$

Now, we want to show that

$$O(b_n^{-2}) = O\left(\left(2\mathrm{lcm}(1, 2, \ldots, n)^3 b_n\right)^{-\alpha}\right),$$

for some $\alpha > 1$. To do this, we will have to understand the asymptotic growth of b_n (and show that it grows faster than our extra multiplier). For large n, the recurrence for b_n approaches

$$b_n = 34 b_{n-1} - b_{n-2}.$$

If we then let the growth rate be exponential, we have

$$r^2 - 34r + 1 = 0,$$

and the root with maximum magnitude here is $17 + 12\sqrt{2} = (1 + \sqrt{2})^4$. On the other hand, it is well-known that

$$\mathrm{lcm}(1, 2, \ldots, n) = e^{n + o(n)}$$

(in fact, this is equivalent to the prime number theorem!). Thus, it simply remains to check that

$$\log\left((1 + \sqrt{2})^4\right) > 3,$$

which is indeed true, so we are done. □

Apery's theorem in and of itself does not have deep number-theoretic consequences, but its method of attack can be useful. In particular, Apery's theorem can be extended to prove that $\zeta(2)$ is irrational, although no one has succeeded in proving that odd zeta values > 3 are irrational. However, it has been shown that the sum of the reciprocals of the Fibonacci numbers is irrational using similar methods.

References

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