Explicit Formulae For $\Psi(z)$ and $\pi(z)$

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Abstract

The asymptotic distribution of primes, first noticed by Carl Friedrich Gauss in the late 18th century, was later proved independently by mathematicians like Hadamard, de la Vallée Poussin, and Erdős, among others. This asymptotic distribution, discussed extensively in the Prime Number Theorem (PNT), states that $\pi(z) \sim \frac{z}{\log(z)}$. However, this theorem does not discuss the number of primes less than finite natural numbers. An exact description of the amount of primes less than any number, known as Riemann's Explicit Formula, was produced in Riemann's 1859 paper "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse". Riemann's explicit formula for the number of primes up to some n is most commonly given by

$$\pi(z) = R(z) + \sum_{\substack{\zeta(p)=0 \\ 0<Re(p)<1}} R(z^p) + \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \int_{z^{\frac{1}{n}}}^{\infty} \frac{1}{x(x^2-1)\log(x)} dx.$$

German mathematician Hans Carl Friedrich von Mangoldt also proved an equivalent formulation of Riemann's explicit formula, stated using the Chebyshev Ψ function:

$$\Psi(z) = z - \sum_{\substack{\zeta(p)=0 \\ 0<Re(p)<1}} \frac{z^p}{p} - \frac{1}{2}\log\left(1 - z^{-2}\right) - \log(2\pi).$$

This paper discusses the various formulations of the one precise formula which gives $\pi(z)$ in terms of a sum over the zeros of the Riemann-ζ function.

1 Hadamard Products and Von Mangoldt's Function

In the late 19th century, German mathematician Karl Weierstraß developed a method for expanding meromorphic functions into products, called the *Weierstraß Factorization Theorem*. The Weierstraß Factorization Theorem is surprisingly intuitive–just as polynomials are defined (by the Fundamental Theorem of Algebra) as a product over their zeroes, all meromorphic functions can be represented almost identically. Building upon this theorem, Jacques Hadamard developed an infinite product representation of the Riemann-ζ function.

Theorem 1.1. (Hadamard)

$$(z-1)\zeta(z) = \frac{1}{2}\Big(\frac{2\pi}{e}\Big)^z \Bigg(\prod_{n=1}^{\infty} e^{\frac{-z}{2n}}\Big(1+\frac{z}{2n}\Big)\Bigg)\Bigg(\prod_{\substack{\zeta(p)=0\\ 0<Re(p)<1}} e^{\frac{z}{p}}\Big(1-\frac{z}{p}\Big)\Bigg).$$

Although this theorem constitutes a significant step in our proof of the Von Mangoldt Explicit Formula, the proof is too sophisticated for this paper, and has been eschewed.

Theorem 1.2.

$$\frac{\zeta'(z)}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{z}{2n(z+2n)} + \sum_{\substack{\zeta(p)=0\\ 0<Re(p)<1}} \frac{z}{p(z-p)} + \log(2\pi) + \frac{z}{1-z}.$$

Proof. Note that

$$\begin{aligned}\frac{\Big((z-1)\zeta(s)\Big)'}{(z-1)\zeta(z)} &= \frac{\zeta(z)+(z-1)\zeta'(z)}{(z-1)\zeta(z)}\\ &= \frac{1}{z-1} + \frac{\zeta'(z)}{\zeta(z)}.\end{aligned}$$

We evaluate the LHS alternately by noting that for some function f, $\frac{f'(z)}{f(z)}$ is always equal to $\frac{d}{dz}\log(f(z))$. Thus, the LHS is equal to

$$\frac{d}{dz}\log\Big((z-1)\zeta(z)\Big).$$

Equating the RHS of the two evaluations and shifting the $\frac{1}{z-1}$ term, we see that

$$\frac{\zeta'(z)}{\zeta(z)} = \frac{d}{dz}\log\Big((z-1)\zeta(z)\Big) - \frac{1}{z-1}.$$

Substituting the Hadamard Product Expansion for $(z-1)\zeta(z)$ into the expression and evaluating the easy terms, we have:

$$\begin{aligned}\frac{\zeta'(z)}{\zeta(z)} &= \frac{d}{dz}\log\Big((z-1)\zeta(z)\Big) - \frac{1}{z-1}\\ &= \frac{d}{dz}\log\Bigg(\frac{1}{2}\Big(\frac{2\pi}{e}\Big)^z\Bigg(\prod_{n=1}^{\infty} e^{\frac{-z}{2n}}\Big(1+\frac{z}{2n}\Big)\Bigg)\Bigg(\prod_{\substack{\zeta(p)=0\\ 0<Re(p)<1}} e^{\frac{z}{p}}\Big(1-\frac{z}{p}\Big)\Bigg)\Bigg) - \frac{1}{z-1}.\\ &= \frac{d}{dz}\Bigg(\log\Big(\frac{1}{2}\Big) + z\log\Big(\frac{2\pi}{e}\Big) + \sum_{n=1}^{\infty}\Big\{e^{\frac{-z}{2n}}\Big(1+\frac{z}{2n}\Big)\Big\} + \sum_{\substack{\zeta(p)=0\\ 0<Re(p)<1}}\Big\{e^{\frac{z}{p}}\Big(1-\frac{z}{p}\Big)\Big\}\Bigg) - \frac{1}{z-1}\\ &= \log(2\pi) + \frac{d}{dz}\Bigg(\sum_{n=1}^{\infty}\Big\{\log\Big(e^{\frac{-z}{2n}}\Big(1+\frac{z}{2n}\Big)\Big)\Big\} + \sum_{\substack{\zeta(p)=0\\ 0<Re(p)<1}}\Big\{\log\Big(e^{\frac{z}{p}}\Big(1-\frac{z}{p}\Big)\Big)\Big\}\Bigg) + \frac{z}{1-z}.\end{aligned}$$

We now tackle the first sum inside the derivative.

$$\sum_{n=1}^{\infty}\left\{\frac{d}{dz}\log\left(e^{\frac{-z}{2n}}\left(1+\frac{z}{2n}\right)\right)\right\} = \sum_{n=1}^{\infty}\left\{\frac{1}{e^{\frac{-z}{2n}}\left(1+\frac{z}{2n}\right)}\left(\frac{-e^{\frac{-z}{2n}}\left(1+\frac{z}{2n}\right)}{2n}+\frac{e^{\frac{-z}{2n}}}{2n}\right)\right\}$$
$$= \sum_{n=1}^{\infty}\left\{\frac{1}{z+2n}-\frac{1}{2n}\right\}.$$

Moving on to the second sum:

$$\sum_{\substack{\zeta(p)=0\\0<Re(p)<1}}\left\{\frac{d}{dz}\log\left(e^{\frac{z}{p}}\left(1-\frac{z}{p}\right)\right)\right\} = \sum_{\substack{\zeta(p)=0\\0<Re(p)<1}}\left\{\frac{d}{dz}\Big(\log\left(e^{\frac{z}{p}}\right)+\log\left(1-\frac{z}{p}\right)\Big)\right\}$$
$$= \sum_{\substack{\zeta(p)=0\\0<Re(p)<1}}\left\{\frac{1}{p}+\frac{1}{z-p}\right\}.$$

Coming back to the original equation with $\frac{\zeta'(z)}{\zeta(z)}$, we see that

$$\frac{\zeta'(z)}{\zeta(z)} = \sum_{n=1}^{\infty}\left\{\frac{1}{z+2n}-\frac{1}{2n}\right\} + \sum_{\substack{\zeta(p)=0\\0<Re(p)<1}}\left\{\frac{1}{p}+\frac{1}{z-p}\right\} + \log(2\pi) + \frac{z}{1-z}.$$

Rewriting the summands with common denominators, we see that

$$\frac{\zeta'(z)}{\zeta(z)} = \sum_{n=1}^{\infty}\frac{z}{2n(z+2n)} + \sum_{\substack{\zeta(p)=0\\0<Re(p)<1}}\frac{z}{p(z-p)} + \log(2\pi) + \frac{z}{1-z}.$$

■

We begin our study of the Von Mangoldt Explicit Formula and the Chebyshev-Ψ function with some more analysis of the function $\frac{\zeta'(z)}{\zeta(z)}$ and the Von Mangoldt-Λ function.

Theorem 1.3.

$$\frac{\zeta'(z)}{\zeta(z)} = -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^z}.$$

Proof. Note that for some function f, we have $\frac{f'(x)}{f(x)} = \frac{d}{dx}\log(f(x))$. It follows from this fact that $\frac{\zeta'(z)}{\zeta(z)}$ should be equal to $\frac{d}{dz}\log(\zeta(z))$. Then, expanding ζ with its corresponding Euler Product, we have:

$$\frac{\zeta'(z)}{\zeta(z)} = \frac{d}{dz}\log\left(\prod_{p\text{ prime}}\left(1-\frac{1}{p^z}\right)^{-1}\right)$$
$$= -\frac{d}{dz}\sum_{p\text{ prime}}\log\left(1-\frac{1}{p^z}\right).$$

With the product rule and a geometric series expansion, the expression becomes:

$$
\begin{aligned}
&= -\sum_{p \text{ prime}} \left(\frac{\log(p)}{p^z}\right)\left(1-\frac{1}{p^z}\right)^{-1} \\
&= -\sum_{p \text{ prime}} \log(p) \sum_{m=1}^{\infty} \frac{1}{p^{mz}} \\
&= -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z}.
\end{aligned}
$$

■

2 The Mellin Transform

Before we proceed any further, we will need to achieve a proper understanding of the *Mellin Transform* and its properties.

Definition 2.1. Mellin Transform

The *Mellin Transform* $\mathcal{M}$ is an operator which alters functions as such: given an explicitly stated function f, the Mellin Transform of f is given by

$$\mathcal{M}\big(f(z)\big) = \int_1^{\infty} x^{z-1} f(x)dx.$$

This integral operator should look very similar to the Gamma Function, because if we let $f(z) = e^{-z}$, then

$$\mathcal{M}\big(f(z)\big) = \int_0^{\infty} x^{z-1} e^{-x} dx = \Gamma(z).$$

Lemma 2.2.

$$\frac{d}{dz}\mathcal{M}\big(f(z)\big) = -\mathcal{M}\big(f \times \log\big)(s).$$

Proof.

$$
\begin{aligned}
\frac{d}{dz}\mathcal{M}\big(f(z)\big) &= \frac{d}{dz}\int_1^{\infty} f(x)x^{-z-1}dx \\
&= \int_1^{\infty} \frac{1}{x} f(x) \frac{d}{dz} x^{-z} dx \\
&= -\int_1^{\infty} f(x)\log(x)x^{-s-1}dx \\
&= -\mathcal{M}\big(f \times \log\big)(s).
\end{aligned}
$$

■

Lemma 2.3. Let $\mathcal{E}$ represent the functional operator defined by $\mathcal{E}\big(f(x)\big) = xf'(x)$, and let $f(1) = 0$. Then, we have

$$\mathcal{M}\Big(\mathcal{E}\big(f(z)\big)\Big) = z\mathcal{M}\big(f(z)\big).$$

Proof.

$$\mathcal{M}\Big(\mathcal{E}\big(f(z)\big)\Big) = \mathcal{M}\big(zf'(z)\big)$$
$$= \int_1^\infty xf'(x)x^{-s-1}dx$$
$$= \int_1^\infty \frac{f'(x)}{x^s}dx.$$

We proceed using integration by parts:

$$\mathcal{M}\Big(\mathcal{E}\big(f(z)\big)\Big) = x^{-z}f(x)\Big|_1^\infty + z\int_1^\infty f(x)x^{-s-1}dx.$$

The first term on the RHS becomes zero; it was assumed in the beginning that $f(1) = 0$, and if f does not approach 0 as $x \to \infty$, then the integral would diverge. So, the RHS of the above equation is left only with the second term, and we have:

$$\mathcal{M}\Big(\mathcal{E}\big(f(z)\big)\Big) = z\int_1^\infty f(x)x^{-s-1}dx.$$

Thus,

$$\mathcal{M}\Big(\mathcal{E}\big(f(z)\big)\Big) = z\mathcal{M}\big(f(z)\big).$$

■

3 The Von Mangoldt Explicit Formula for Ψ

Now that we have discussed the Mellin Transform in sufficient depth, we will proceed to Von Mangoldt's Explicit Formula. The *Von Mangoldt Explicit Formula* is an exact formula for the Chebyshev-Ψ Function in terms of sums over the zeroes of the Riemann-ζ Function.

Definition 3.1. The Chebyshev-Ψ Function

We define $\Psi(z)$ as

$$\Psi(z) = \sum_{n\le z} \Lambda(n).$$

Before introducing and proving the Von Mangoldt Explicit formula, we must first establish some groundwork.

Lemma 3.2.

$$z\int_n^\infty x^{-z-1}dx = n^{-z}.$$

Proof.

$$z\int_n^\infty x^{-z-1}dx = -x^{-z}\Big|_n^\infty$$
$$= 0 - (-n^{-z})$$
$$= n^{-z}.$$

■

Lemma 3.3.
$$\frac{\zeta'(z)}{\zeta(z)} = -z\mathcal{M}\big(\Psi(z)\big).$$

Proof. It follows from Lemma 3.2 that
$$-\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^z} = -\sum_{n=1}^{\infty}\Lambda(n)z\int_n^{\infty}x^{-z-1}dx.$$
According to Theorem 1.3, the LHS is equal to $\frac{\zeta'(z)}{\zeta(z)}$, so we have
$$\frac{\zeta'(z)}{\zeta(z)} = -\sum_{n=1}^{\infty}\Lambda(n)z\int_n^{\infty}x^{-z-1}dx.$$
Because the variable x is always greater than or equal to n, we can swap the sum and the integral to produce
$$\frac{\zeta'(z)}{\zeta(z)} = -z\int_1^{\infty}\sum_{n\leq x}\Lambda(n)x^{-z-1}dx.$$
We now replace the sum over the Von Mangoldt-Λ Function with the Chebyshev-Ψ Function:
$$\begin{aligned}\frac{\zeta'(z)}{\zeta(z)} &= -z\int_1^{\infty}\Psi x^{-z-1}dx\\ &= -z\mathcal{M}\big(\Psi(z)\big).\end{aligned}$$
■

Theorem 3.4. The Von Mangoldt Explicit Formula for $\Psi(z)$
$$\Psi(z) = z - \sum_{\substack{\zeta(p)=0\\ 0<Re(p)<1}}\frac{z^p}{p} - \frac{1}{2}\log\left(1-z^{-2}\right) - \log(2\pi).$$

Proof. We know from Lemma 3.3 that
$$\frac{\zeta'(z)}{\zeta(z)} = -z\mathcal{M}\big(\Psi(z)\big).$$
According to Theorem 1.2, the LHS of this equation can be rewritten:
$$\log(2\pi) + \sum_{n=1}^{\infty}\frac{z}{2n(z+2n)} + \sum_{\substack{\zeta(p)=0\\ 0<Re(p)<1}}\frac{z}{p(z-p)} + \frac{z}{1-z} = -z\mathcal{M}\big(\Psi(z)\big).$$
We can now divide by $-z$ and inverse Mellin Transform on both sides. This produces the exact formula for Ψ which we want:
$$\Psi(z) = z - \sum_{\substack{\zeta(p)=0\\ 0<Re(p)<1}}\frac{z^p}{p} - \frac{1}{2}\log\left(1-z^{-2}\right) - \log(2\pi).$$
It is possible to check this result by completing a term-by-term Mellin Transform on Von Mangoldt's Explicit Formula and multiplying by $-z$, which would give us the RHS of Theorem 1.2. ■

4 The Riemann Explicit Formulae for $\Pi(z)$ and $\pi(z)$

Definition 4.1. The Prime Counting Function

We define $\pi(z)$ to be the number of primes less than z. Written mathematically, this may appear as

$$\pi(z) = \sum_{\substack{p<z \\ p \text{ prime}}} 1.$$

Definition 4.2. The Π-Function

We define $\Pi(z)$ with the sum:

$$\Pi(z) = \sum_{n=1}^{\infty} \frac{\pi(z^{\frac{1}{n}})}{n} = \pi(z) + \frac{1}{2}\pi(z^{\frac{1}{2}}) + \frac{1}{3}\pi(z^{\frac{1}{3}})\ldots$$

Let the reader note that $\Pi(1)$ is equal to 0, since the function $\pi(z)$ within the sum takes a value of zero when $z = 1$.

Definition 4.3. The Logarithmic Integral

We define the *Logarithmic Integral* function $\text{Li}(z)$ to be:

$$\text{Li}(z) = \int_2^z \frac{1}{\log(x)} dx.$$

Theorem 4.4. Riemann's Explicit Formula for $\Pi(z)$

Riemann's Explicit Formula for $\Pi(z)$ is as follows for $z > 1$:

$$\Pi(z) = \text{Li}(z) - \sum_{\substack{\zeta(p)=0 \\ 0<Re(p)<1}} \text{Li}(z^p) - \log(2) + \int_z^{\infty} \frac{1}{x(x^2-1)\log(x)} dx.$$

Proof. **(Part 1)** We wish to show in Part 1 of this proof that $\log\big(\zeta(z)\big) = z\mathcal{M}\big(\Pi(z)\big)$. We begin by taking the logarithm of the Riemann Zeta Function:

$$\begin{aligned} \log\big(\zeta(z)\big) &= \log\left(\prod_{p\,\text{prime}} \left(1 - \frac{1}{p^z}\right)^{-1}\right) \\ &= -\sum_{p\,\text{prime}} \log\left(1 - \frac{1}{p^z}\right). \end{aligned}$$

Knowing the Taylor series expansion of $\log(1-x)$, it is possible to expand this sum into

$$\log\big(\zeta(z)\big) = \sum_{p \text{ prime}} \sum_{n=1}^{\infty} \frac{1}{n} p^{-nz}.$$

Then, using Lemma 3.2, we find that

$$\log\big(\zeta(z)\big) = \sum_{p \text{ prime}} \sum_{n=1}^{\infty} \frac{z}{n} \int_{p^n}^{\infty} x^{-z-1} dx.$$

Exchanging the order of the sum and the integral, we have

$$\log\big(\zeta(z)\big) = z\int_1^\infty \sum_{\substack{p,n \\ p^n<x}} \frac{1}{n} x^{-z-1}dx.$$

Recall that $\Pi(x)$ is equivalent to $\sum_{\substack{p,n \\ p^n<x}} \frac{1}{n}$ except for a set of points with measure zero. This implies that $\Pi(z)$ can substitute the sum in question without changing the value of the integral. Then, we find that

$$\log\big(\zeta(z)\big) = z\int_1^\infty \Pi(x)x^{-z-1}dx.$$

Thus,

$$\log\big(\zeta(z)\big) = z\mathcal{M}\big(\Pi(z)\big).$$

(Part 2) We proceed with the derivation of Riemann's Explicit Formula for $\Pi(z)$ in the second portion of this proof. Note that $\Psi(1) = 0$; this allows the use of Lemma 2.3, which is used to show that

$$\mathcal{M}\Big(\mathcal{E}\big(\Psi(z)\big)\Big) = z\mathcal{M}\big(\Psi(z)\big).$$

Employing Lemma 3.3, we find that

$$\mathcal{M}\Big(\mathcal{E}\big(\Psi(z)\big)\Big) = -\frac{\zeta'(z)}{\zeta(z)}.$$

Since the RHS is equal to negative one multiplied with the logarithmic derivative of $\zeta(z)$, we have:

$$\mathcal{M}\Big(\mathcal{E}\big(\Psi(z)\big)\Big) = -\frac{d}{dz}\log\big(\zeta(z)\big).$$

Recall that it was shown in part 1 of this proof that $\log\big(\zeta(z)\big) = z\mathcal{M}\big(\Pi(z)\big)$. Then,

$$\mathcal{M}\Big(\mathcal{E}\big(\Psi(z)\big)\Big) = -\frac{d}{dz}z\mathcal{M}\big(\Pi(z)\big).$$

Then, by the product rule,

$$\mathcal{M}\Big(\mathcal{E}\big(\Psi(z)\big)\Big) = -\mathcal{M}\big(\Pi(z)\big) - z\frac{d}{dz}\mathcal{M}\big(\Pi(z)\big).$$

By Lemma 2.2, we have:

$$\mathcal{M}\Big(\mathcal{E}\big(\Psi(z)\big)\Big) = -\mathcal{M}\big(\Pi(z)\big) + z\mathcal{M}\big(\Pi \times \log\big)(z).$$

The fact that $\Pi(1) = 0$ allows us to employ Lemma 2.3 again, which then allows us to write

$$\mathcal{M}\Big(\mathcal{E}\big(\Psi(z)\big)\Big) = -\mathcal{M}\big(\Pi(z)\big) + \mathcal{M}\Big(\mathcal{E}\big(\Pi \times \log\big)(z)\Big).$$

Let the reader note that the Mellin Transform is *injective*, or *one-to-one*, which then implies that the Mellin Transform is distinctly invertible. This, then, allows us to operate an inverse Mellin Transform on both sides of the above equation, in order to produce the following equality:

$$\mathcal{E}\big(\Psi(z)\big) = -\Pi(z) + \mathcal{E}\big(\Pi \times \log\big)(z).$$

All that is left to do is check that when each term from Riemann's Explicit Formula for $\Pi(z)$ is substituted into $-\Pi(z) + \mathcal{E}\big(\Pi \times \log\big)(z)$, it produces its respective term in $\mathcal{E}\big(\Psi(z)\big)$, where $\Psi(z)$ is expanded using the Von Mangoldt Explicit Formula discussed in Theorem 3.2. This completes the proof. ■

Theorem 4.5.

$$\pi(z) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \Pi(z^{\frac{1}{n}}).$$

Proof. Recall that we defined $\Pi(z)$ to be

$$\begin{aligned}\Pi(z) &= \pi(z) + \frac{1}{2}\pi(z^{\frac{1}{2}}) + \frac{1}{3}\pi(z^{\frac{1}{3}}) \ldots \\ &= \sum_{n=1}^{\infty} \frac{\pi(z^{\frac{1}{n}})}{n}.\end{aligned}$$

Then, by the Möbius Inversion, we have:

$$\begin{aligned}\pi(z) &= \Pi(z) - \frac{1}{2}\Pi(z^{\frac{1}{2}}) - \frac{1}{3}\Pi(z^{\frac{1}{3}}) - \frac{1}{5}\Pi(z^{\frac{1}{5}}) + \frac{1}{6}\Pi(z^{\frac{1}{6}}) \ldots \\ &= \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \Pi(x^{\frac{1}{k}}).\end{aligned}$$

■

Equipped with an equation relating $\pi(z)$ and $\Pi(z)$ and Riemann's Explicit Formula for $\Pi(z)$, we can now develop the explicit formula for $\pi(z)$. Before doing so, however, it is useful to simplify lengthy functions into new ones, which provides the motivation for the R-Function.

Definition 4.6. The R-Function

Let R be defined as such:

$$\begin{aligned}R(z) &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{Li}(z^{\frac{1}{n}}) \\ R(z^p) &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{Li}(z^{\frac{p}{n}}).\end{aligned}$$

Now, it is possible to rewrite Riemann's Explicit Formula in a way that shows precisely how many primes exist below a certain z.

Theorem 4.7. Riemann's Explicit Formula

$$\pi(z) = R(z) + \sum_{\substack{\zeta(p)=0 \\ 0<Re(p)<1}} R(z^p) + \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \int_{z^{\frac{1}{n}}}^{\infty} \frac{1}{x(x^2-1)\log(x)} dx.$$

Proof. This proof follows from the Möbius Inversion relationship between $\pi(z)$ and $\Pi(z)$ provided in Theorem 4.5, and from Riemann's Explicit Formula for $\Pi(z)$, provided in

Theorem 4.4. By substituting Riemann's Explicit Formula for $\Pi(z)$ into the result from Theorem 4.5, we see that

$$\pi(z) = R(z) + \sum_{\substack{\zeta(p)=0 \\ 0<Re(p)<1}} R(z^p) + \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \int_{z^{\frac{1}{n}}}^{\infty} \frac{1}{x(x^2-1)\log(x)} dx.$$

■

5 Consequences of Riemann's Explicit Formula

Theorem 5.1. Riemann's Explicit Formula implies the Prime Number Theorem.

The explanation provided is by no means a rigorous proof. Instead, we build an intuition as to why the explicit formulae mentioned in this paper imply the PNT. Note that there are many formulations of the Prime Number Theorem, the most common of which is

$$\pi(z) \sim \frac{z}{\log(z)}.$$

This is read as $\pi(z)$ is *asymptotic* to $\frac{z}{\log(z)}$, which means that

$$\lim_{z\to\infty} \frac{\pi(z)}{z/\log(z)} = 1.$$

While this representation of the Prime Number Theorem is useful to know, we will be working with a different version of the PNT, which says that

$$\pi(z) \sim \mathrm{Li}(z),$$

where $\mathrm{Li}(z)$ is the *Logarithmic Integral* discussed in Definition 4.3. We provide a short proof of this version of the PNT below.

Proof. The Logarithmic Integral has a commonly known series expansion. This expansion can be derived using integration by parts. Recall that $\mathrm{Li}(z)$ is defined to be

$$\mathrm{Li}(z) = \int_2^z \frac{1}{\log(x)} dx.$$

We integrate the function by parts to produce

$$\int \frac{1}{\log(x)} dx = \frac{x}{\log(x)} + \int \frac{1}{\log^2(x)} dx.$$

Then,

$$\begin{aligned}\mathrm{Li}(z) &= \frac{x}{\log(x)}\bigg|_2^z + \int_2^z \frac{1}{\log^2(x)} dx. \\ &= \frac{z}{\log(z)} + \int_2^z \frac{1}{\log^2(x)} dx + O(1).\end{aligned}$$

Thus,

$$\mathrm{Li}(z) = O\left(\frac{z}{\log(z)}\right).$$

Recall that the prime counting function $\pi(z)$ has similar asymptotics. It is already given by the most commonly known version of the PNT that

$$\pi(z) = O\left(\frac{z}{\log(z)}\right).$$

The identical asymptotics of both the Logarithmic Integral and $\pi(z)$ then imply that

$$\pi(z) \sim \mathrm{Li}(z).$$

■

To understand how Riemann's Explicit Formulae imply this version of the Prime Number Theorem, we must examine the explicit formula for $\Pi(z)$ from Theorem 4.4. The most significant term, or the one that dominates all others, is the $\mathrm{Li}(z)$ term. This implies that $\Pi(z) \sim \mathrm{Li}(z)$. This is not, however, the only asymptotic we can produce for $\Pi(z)$. Given the series expansion for $\Pi(z)$ expressed in Definition 4.2, we see that the first term $\pi(z)$ is by far the most significant. Then, $\Pi(z) \sim \pi(z)$. By the transitive property, we can conclude that $\pi(z) \sim \mathrm{Li}(z)$. Thus, Riemann's Explicit Formulae for $\Pi(z)$ and $\pi(z)$ imply the Prime Number Theorem. ■

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