THE WIENER-IKEHARA THEOREM AND THE PRIME NUMBER THEOREM

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1. Introduction and preliminary notions

In this paper, we give a complex-analytic proof of the prime number theorem, following Korevaar closely. We first show that $\psi(n) \sim n$ is equivalent to the prime number theorem, then show the relation itself, and finally we prove a special case of the Wiener-Ikahara theorem and derive the relation from it.

Definition 1.1 (Dirichlet Series)**.** A *Dirichlet series* is a series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where s is complex and a_n is a sequence of complex numbers.

Theorem 1.2 (Wiener-Ikahara)**.** *Suppose that the Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z}, \quad \textit{with coefficients } a_n \geq 0,$$

converges on the half-plane $\{\Re z > 1\}$*. The sum function* $f(z)$ *is analytic in that open half-plane, so suppose that there is a constant* A *such that the difference*

$$g(z) = f(z) - \frac{A}{z-1}$$

has an analytic or continuous extension to the closed half-plane $\{\Re z \geq 1\}$*. Also, suppose that there is a constant* C *such that* $s_n = \sum_{k \leq n} a_k \leq Cn$ *for all* n*. Then*

$$s_n \sim An \quad \textit{as } n \to \infty.$$

Equivalently, $s_n/n \to A$*.*

Definition 1.3. Let

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and integer } k \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Definition 1.4. Let

$$\psi(x) = \sum_{n \leq x}^{x} \Lambda(n).$$

Theorem 1.5. *For* $\psi(n)$ *defined as above,*

$$\psi(n) \sim n.$$

Theorem 1.6. *Theorem 1.5 is equivalent to the prime number theorem.*

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2. Proving Theorem 1.6

We begin with a lemma.

Lemma 2.1. $\psi(x) \sim \pi(x)\log x$

Proof. First, we show that $\psi(x) \le \pi(x)\log x$. Note that we have

$$\psi(x) = \sum_{p\le x} \log p \left\lfloor \frac{\log x}{\log p} \right\rfloor \le \sum_{p\le x} \log(x) = \pi(x)\log x.$$

For the other direction, let $\epsilon > 0$. Then we have

$$\psi(x) \ge \sum_{x^{1-\epsilon}\le p\le x} \log p \ge \sum_{x^{1-\epsilon}\le p\le x} (1-\epsilon)\log x = (1-\epsilon)(\pi(x) + O(x^{1-\epsilon}))\log x.$$

As this holds for arbitrarily small ϵ, it follows that $\psi(x) \sim \pi(x)\log x$. ■

Now assume Theorem 1.5. We get $x \sim \pi(x)\log x$, or $\pi(x) \sim \frac{x}{\log x}$, as desired. Now assume PNT. We have $\psi(x) \sim \pi(x)\log x$, or $\psi(x) \sim \frac{x}{\log x}\log x \sim x$, as we wanted. This proves 1.6

3. Proving Theorem 1.5

We want to use Theorem 1.2 with $A = 1$, $s_n = \psi(n)$, and $a_n = \Lambda(n)$. In order to get there, we begin with the zeta function, which can be represented by its Euler product:

$$\zeta(z) = \prod_{p \text{ prime}} \left(1 + \frac{1}{p^z} + \frac{1}{p^{2z}} + \cdots\right) = \prod_{p \text{ prime}} \frac{1}{1-p^{-z}}$$

Taking the logarithmic derivative of this gives

$$\begin{aligned}
\frac{\zeta(s)'}{\zeta(s)} &= \left(\log \prod_{p \text{ prime}} (1-p^{-z})^{-1}\right)' \\
&= \left(\sum_{p \text{ prime}} \log(1-p^{-z})^{-1}\right)' \\
&= -\sum_{p \text{ prime}} \big(\log(1-p^{-z})\big)' \\
&= -\sum_{p \text{ prime}} \frac{p^{-z}\log p}{1-p^{-z}} \\
&= \sum_{p \text{ prime}} \frac{\log p}{1-p^{z}} \\
&= \sum_{p \text{ prime}, m\ge 1} \frac{\log p}{p^{mz}} \\
&= \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z},
\end{aligned}$$

where the last equality holds since $\Lambda(n) = 0$ whenever n is not a power of p. Hence, let

$$f_1(z) = \frac{\zeta(s)'}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z}$$

In order to use 1.2, we need to be able to extend $f_1(z)$ to a function $g_1(z)$ that is also analytic on the line $\{\Re(z) = 1\}$. We will assume the following lemma:

Lemma 3.1. *$\zeta(z) \neq 0$ on the line $\{\Re(z) = 1\}$, except for $z = 1$.*

It follows that $\frac{-\zeta(z)'}{\zeta(z)}$ is analytic on $\{\Re(s) = 1\}$ except for at $s = 1$. For $s = 1$ we use the following lemma:

Lemma 3.2. *As $s \to 1$, $\zeta(s) \sim \frac{1}{s-1}$.*

Proof. We apply the following slightly modified form of Euler's Summation Formula:
Let $\phi(x)$ be any function that is differentiable with a continuous derivative on the closed interval $[a, b]$. Then we have that $\sum_{a<n\leq b} \phi(n)$ is equal to the following:

$$\int_a^b \phi(x)dx + \int_a^b \left(x - \lfloor x \rfloor - \frac{1}{2}\right) \phi'(x)dx + \left(a - \lfloor a \rfloor - \frac{1}{2}\right) \phi(a) - \left(b - \lfloor b \rfloor - \frac{1}{2}\right) \phi(b).$$

The proof of this result is left to the reader.
Applying this to the function $\phi(x) = x^{-s}$ yields

$$\sum_{n=a+1}^{b} = \frac{b^{1-s} - a^{1-s}}{1-s} - s \int_a^b \frac{x - \lfloor x \rfloor - \frac{1}{2}}{x^{s+1}} dx + \frac{1}{2}\left(\frac{1}{b^s} - \frac{1}{a^s}\right).$$

Letting $a = 1$, $b \to \infty$, adding 1 to both sides, and assuming that $\mathcal{R}(s) > 1$, yields $\zeta(s)$, so

$$\zeta(s) = \frac{1}{s-1} + s \int_1^\infty \frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{s+1}} dx + \frac{1}{2}.$$

The result follows. ■

Hence, the function $g_1(z) = f_1(z) - \frac{1}{z-1}$ is analytic on $\Re(s) \geq 1$.
We now need a bound C such that $s_n = \psi(n) \leq Cn$ for all n. We obtain this using Chebyshev's inequality:

Lemma 3.3. *There exists a constant C such that $\pi(n) \leq C\frac{n}{\log n}$*

Proof. Note that $n^{\pi(2n)-\pi(n)} \leq \prod_{n<p\leq 2n} \leq \binom{2n}{n} \leq 2^{2n}$, as every prime p with $n < p \leq 2n$ appears once in $2n!$ but never in $n!$. Taking $\log_n$ on both sides, we get $\pi(2n) \leq \pi(n) + 2\log 2\frac{n}{\log n}$. Using induction, it is easy to show that $\pi(2^k) \leq 3 \cdot \frac{2^k}{k}$: by the previous inequality, we have that when $k \geq 5$,

$$\pi(2^{k+1}) \leq \pi(2^k) + \frac{2^{k+1}}{k} \leq 3 \cdot \frac{2^k}{k} + 2 \cdot \frac{2^k}{k} = 5 \cdot \frac{2^k}{k} \leq 3 \cdot \frac{2^{k+1}}{k+1}.$$

But $\frac{x}{\log x}$ is a monotonically increasing function, so $4 \leq 2^k < x \leq 2^{k+1}$ implies

$$\pi(x) \leq \pi(2^{k+1}) \leq 6 \cdot \frac{2^k}{k+1} \leq 6 \cdot \frac{2^k}{k-1} = 6\log 2\frac{2^k}{\log 2^k} \leq 6\log\frac{x}{\log x}$$

It's easy to check that $\pi(x) \leq 6\log 2\frac{x}{\log x}$ when $x \leq 4$, so the proof is complete. ■

Using Lemma 3.3, we have

$$\psi(n) = \sum_{p \leq n} \left(\frac{\log n}{\log p} \right) \log p = \log n \sum_{p \leq n} 1 = \pi(n) \log(n) \leq Cn.$$

Thus, with $A = 1$, $s_n = \psi(n)$, and $a_n = \Lambda(n)$, the assumptions of Theorem 1.2 hold. It follows then that $\psi(n) \sim n$, as desired.

4. Proof of Theorem 1.2

Assume the conditions of Theorem 1.2 hold. Let

$$s(v) = \sum_{k \leq v} a_k,$$

which means that $s(v) = s_n$ when $n \leq v < n + 1$ and $s(v) = 0$ when $v < 1$. Now, partial summing 1.2 gives

$$f(z) = \sum_{n=1}^{\infty} \frac{s_n - s_{n-1}}{n^z}.$$

But note that each s_m will be multiplied first by $\frac{1}{m^z}$ when $n = m$, and then by $-\frac{1}{(m+1)^{z+1}}$ when $n = m + 1$, and so we can rewrite the sum as

$$\sum_{n=1}^{\infty} s_n \left(\frac{1}{n^z} - \frac{1}{(n+1)^z} \right).$$

By inserting a factor of z, we can rewrite the summand as an integral, and then combine the integrals to obtain $f(z)$ as an integral in terms of z:

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} s_n z \int_n^{n+1} v^{-z-1} dv \\ &= z \int_1^{\infty} s(v) v^{-z-1} dv \end{aligned}$$

Using this, we can rewrite $g(z) - A$ as:

$$\begin{aligned} g(z) - A &= f(z) - \frac{A}{z-1} - A \\ &= f(z) - \frac{Az}{z-1} \\ &= z \left(\int_1^{\infty} s(v) v^{-z-1} dv - \int_1^{\infty} A v^{-z} dv \right) \\ &= z \int_1^{\infty} \left(\frac{s(v)}{v} - A \right) v^{-z} dv \end{aligned}$$

Now substitute

$$v = e^t, \frac{s(v)}{v} - A = e^{-t} s(e^t) = \rho(t),$$

and let $\rho(t) = 0$ when $t < 0$. But for $t > u \geq 0$, we have $s(e^t) \geq s(e^u)$, and so

$$\begin{aligned}
\rho(t) - \rho(u) &= e^{-t}s(e^t) - e^{-u}s(e^u) \\
&\geq (e^{-t} - e^{-u})s(e^u) \\
&= (e^{-(t-u)} - 1)e^{-u}s(e^u) \\
&= -(1 - e^{-(t-u)})(\rho(u) + A) \\
&\geq -(t-u)(\rho(u) + A).
\end{aligned}$$

As a function that satisfies $f(t) - f(u) \geq -\mu(t,u)$ when $\mu(t,u) \to 0$ as $u \to \infty$ and $0 < t - u \to 0$ is called *slowly decreasing*, ρ is slowly decreasing. Now consider the following Laplace transform of ρ:

$$\begin{aligned}
\mathcal{L}\rho(z) &= \int_0^\infty \rho(t)e^{-zt}dt \\
&= \int_0^\infty \left(\frac{s(v)}{v} - A\right) v^{-z-1}dv \\
&= \frac{g(z+1) - A}{z+1}.
\end{aligned}$$

Hence, we have to prove that as $t \to \infty$, $\rho(t) \to 0$.

The following theorem will suffice then:

Theorem 4.1. *Let $\rho(t) = 0$ for $t < 0$ and $|\rho(t)| \geq M < \infty$ for $t \geq 0$. Then the Laplace transform*

$$G(z) = \mathcal{L}\rho(z) = \int_0^\infty \rho(t)e^{-zt}dt, z = x + yi$$

defines an analytic function for $x > 0$. Suppose that for $-R \leq y \leq R$, the function $g(x+yi)$ converges uniformly to a limit function as x approaches 0 from the positive side. Then for every positive T and δ,

$$\left|\int_T^{T+\delta} \rho(t)dt\right| \leq \frac{4M}{R} + \frac{1}{2\pi}\left|\int_{-R}^{R} G(iy)\frac{e^{i\delta y} - 1}{y}\left(1 - \frac{y^2}{R^2}\right)e^{iTy}dy.\right|$$

If R can be arbitrarily large, and ρ is slowly decreasing, then

$$\rho(T) \to 0 \text{ as } T \to \infty.$$

5. Proof of 4.1

Proof. Let $G_T(z) = \int_0^T \rho(t)e^{-zt}dt$, where $z = x + yi$. Our goal is to estimate the difference

$$G_{T+\delta}(0) - G_T(0) = \int_T^{T+\delta} \rho(t)dt.$$

Let Γ be the positively oriented circle $C(0,R) = \{|z| = R\}$. Using Cauchy's formula, we have

$$2\pi i G_T(0) = \int_\Gamma G_T(z)\frac{1}{z}dz.$$

When T is large, and z is in the right half of the plane ($\Re(z) > 0$, we have

$$|G_T(z) - G(z)| = \left|\int_T^\infty \rho(t)e^{-zt}dt\right| \leq M\int_T^\infty e^{-xt}dt = \frac{M}{x}e^{-Tx}$$

Similarly, one can find

$$|G_T(z) = \left|\int_0^T \rho(t)e^{-zt}dt\right| \leq \int_0^T e^{-xt}dt < \frac{M}{|x|}e^{-Tx}.$$

If we replace the $\frac{1}{z}$ in the Cauchy formula with $\frac{1}{z} + \frac{z}{R^2}$ and insert a factor of e^{Tz} into the integrand, we get

$$2\pi i G_T(0) = \int_\Gamma G_T(z)e^{Tz}\left(\frac{1}{z} + \frac{z}{R^2}\right)dz$$

by the residue theorem.

Now we want to do something similar but with $G(z)$. The problem is that $G(z)$ need not be analytic when $\Re(z) < 0$, and so we can't integrate over the entirety of Γ. Instead, we apply Cauchy's theorem to

$$G(z)e^{Tz}\left(\frac{1}{z} + \frac{z}{R^2}\right)$$

over a path that lies entirely in the right half-plane. In order to do this, we introduce some paths of integration. Let Γ_1 and Γ_2 be the parts of Γ in the right and left half-planes, respectively. Let σ be the oriented segment of the imaginary axis from iR to $-iR$. Finally, when r is small let σ_r be the oriented segment of the line $x = r$ between $z_1 = r + \sqrt{R^2 - r^2}$ and $z_2 = r - i\sqrt{R^2 - r^2}$ on Γ. We refer to this part of Γ_1 as $\Gamma_{1,r}$. By Cauchy's theorem, then, we have

$$0 = \int_{\Gamma_{1,r}+\sigma_r} G(z)e^{Tz}\left(\frac{1}{z} + \frac{z}{R^2}\right)dz.$$

Combining the above expressions, and assuming that $r > 0$ is small, we get that

$$2\pi i G_T(0) = I_1(R, r, T) + I_2(R, r, T) - I_3(R, r, T),$$

where

$$I_1(R, r, T) = \int_{\Gamma_{1,r}} (G_T(z) - G(z))e^{Tz}\left(\frac{1}{z} + \frac{z}{R^2}\right)dz$$

$$L_2(R, r, T) = \int_{\Gamma-\Gamma_{1,r}} G_T(z)e^{Tz}\left(\frac{1}{z} + \frac{z}{R^2}\right)dz$$

$$L_3(R, r, T) = \int_{\sigma_r} G_T(z)e^{Tz}\left(\frac{1}{z} + \frac{z}{R^2}\right)dz$$

There is an analogous formula for $G_{T+\delta}(z)$. Since we seek to estimate $G_{T+\delta}(z) - G(z)$, note that

$$\begin{aligned} I_3(R, r, T+\delta) - I_3(R, r, T) &= \int_{\sigma_r} G(z)(e^{(T+\delta)z} - e^{Tz})\left(\frac{1}{z} + \frac{z}{R^2}\right)dz \\ &= \int_{\sigma_r} G(z)\frac{e^{\delta z} - 1}{z}\left(1 + \frac{z^2}{R^2}\right)e^{Tz}dz \end{aligned}$$

As we assume that $G(x+yi)$ converges uniformly as x approaches 0, we can let r go to 0 in this expression. We can also do the same for the corresponding expression for $G_{T+\delta}(0) - G_T(0)$.

Let $I_j(R,0,\cdot)$ be the integrals with $\Gamma_{1,r}$ replaced with Γ_1 and σ_r replaced with σ. Putting these expressions together, we get

$$2\pi|G_{T+\delta}(0)-G_T(0)| \leq$$
$$I_1(R,0,T+\delta)-I_1(R,0,T)+I_2(R,0,T+\delta)-I_2(R,0,T)|+\int_\delta G(z)\frac{e^{\delta z}}{z}\left(1+\frac{z^2}{R^2}\right)e^{Tz}dz.$$

From above, we can derive that

$$\begin{aligned}|I_1(R,0,T)| &\leq \int_{\Gamma_1}|G_T(z)-G(z)|e^{Tz}|\left|\frac{1}{z}+\frac{z}{R^2}\right| \\ &\leq \int_{\Gamma_1}\frac{M}{x}e^{-Tx}e^{Tx}\frac{2x}{R^2}|dz| \\ &= \frac{2M}{R^2}\pi R \\ &= \frac{2\pi M}{R}\end{aligned}$$

We also have

$$\begin{aligned}|I_2(R,0,T)| &\leq \int_{\Gamma_2}|G_T(z)e^{Tz}\left|\frac{1}{z}+\frac{z}{R^2}\right||dz| \\ &= \frac{2\pi M}{R}\end{aligned}$$

Substituting the last few expressions into our desired difference and dividing by 2π we obtain

$$|\int_T^{T+\delta}| \leq \frac{4M}{R}+\frac{1}{2\pi}\int_\sigma G(z)\frac{e^{\delta z}}{z}\left(1+\frac{z^2}{R^2}\right)e^{Tz}dz.$$

Substituting $z=iy$ for $-R\leq y\leq R$ immediately gives the desired bound. It suffices then to show that $\rho(t)\to 0$ as $t\to 0$ under the assumptions listed.

6. $\rho(t)\to 0$ as $t\to 0$

If the $G(iy)$ are continuous or integrable, we use the Riemann-Lebesgue lemma, which tells us that the integral vanishes at infinity. If $G(z)$ is analytic on the segment $[-iR,iR]$, then integration by parts gives $e^{iTy}dy=(\frac{1}{iT})de^{iTy}$. In either case, we get

$$\limsup_{T\to\infty}\left|\int_T^{T+\delta}\rho(t)dt\right| \leq \frac{4M}{R}$$

Now suppose R is arbitrarily large. Then for every $\delta>0$, as $T\to\infty$ we have

$$\int_T^{T+\delta}\rho(t)dt\to 0$$

Finally, assume that $\rho(t)$ is slowly decreasing. Then integrating both sides of $\rho(t)-\rho(u)\geq -\eta(t,T)$ from T to $T+\delta$ and using the above shows that $\limsup_{T\to\infty}\delta\rho(T)\leq\epsilon\delta$ and $\limsup_{T\to\infty}\rho(T)\leq\epsilon$. Hence, $\limsup\rho(T)\leq 0$. Using say $\int_{T-\delta}^T\rho(t)dt$, we can obtain an inequality in the other direction, finishing the proof. ■

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