POLYLOGARITHMS AND MULTIPLE ZETA VALUES

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Abstract. In this paper, we explore variations of the polylogarithm function as well as their properties. In addition, we examine the multiple zeta function and its similarities with the multiple polylogarithm.

We begin with an introduction to polylogarithms, including their definitions and basic identities. Then, we look at the dilogarithm in particular, proving some of its functional equations. Finally, we introduce the multivariable versions of both the polylogarithm and Riemann zeta function and observe the relations between the two.

1. Introduction

The polylogarithm function is a special function $\mathrm{Li}_k(x)$ of order k and argument x. For some values of k, it can be expressed in terms of elementary functions. The function comes up often in fields such as quantum statistics.

Definition 1.1. The *polylogarithm function* is a power series in x and a Dirichlet series in k defined for $|x| < 1$ (but can be analytically extended to a larger domain) by

$$\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}.$$

There are a plethora of functional equations with regard to the general polylogarithm as well as specific forms of it. One important one is the square relationship.

Proposition 1.2 (Square Relationship)**.** *We have the following duplication formula:*

$$\mathrm{Li}_k(-x) + \mathrm{Li}_k(x) = 2^{1-k}\,\mathrm{Li}_k(x^2).$$

Proof. This can be easily shown by adding the series expanisions of the left hand side and eliminating the zero terms.

$$\begin{array}{ccccccccccccccccc}
\mathrm{Li}_k(x) & = & x & + & \frac{x^2}{2^k} & + & \frac{x^3}{3^k} & + & \frac{x^4}{4^k} & + & \frac{x^5}{5^k} & + & \frac{x^6}{6^k} & + & \frac{x^7}{7^k} & + & \cdots \\
\mathrm{Li}_k(-x) & = & -x & + & \frac{x^2}{2^k} & - & \frac{x^3}{3^k} & + & \frac{x^4}{4^k} & - & \frac{x^5}{5^k} & + & \frac{x^6}{6^k} & - & \frac{x^7}{7^k} & + & \cdots \\
\hline
\mathrm{Li}_k(x) + \mathrm{Li}_k(-x) & = & & & \frac{x^2}{2^{k-1}} & & & + & \frac{x^4}{2^{k-1}.2^k} & & & + & \frac{x^6}{2^{k-1}.3^k} & & & + & \cdots
\end{array}$$

Note that the bottom line is $2^{1-k}\,\mathrm{Li}_k(x^2)$, so, as desired, we have

$$\mathrm{Li}_k(-x) + \mathrm{Li}_k(x) = 2^{1-k}\,\mathrm{Li}_k(x^2).$$

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For some values of k, $\mathrm{Li}_k(x)$ can be expressed neatly in closed form. These values include

$$\begin{aligned}\mathrm{Li}_1(x) &= -\log(1-x),\\ \mathrm{Li}_0(x) &= \frac{x}{1-x},\\ \text{and } \mathrm{Li}_{-1}(x) &= \frac{z}{(1-z)^2}.\end{aligned}$$

Furthermore, the closed form expressions for all negative integer values of k can be easily found using the recurrence formula, given below.

Proposition 1.3 (Recurrence Relation)**.** *The recurrence formula shows the relationship between lower- and higher-order polylogarithms, and can be stated in either of the two following forms:*

$$\mathrm{Li}_{k+1}(x) = \int_0^x \frac{\mathrm{Li}_k(t)}{t}\,dt, \qquad x\,\mathrm{Li}'_k(x) = \mathrm{Li}_{k-1}(x)$$

Proof. The formula can be shown by differentiating the series term-by-term.

$$\begin{aligned}\mathrm{Li}_k(x) &= \sum_{n=1}^{\infty} \frac{x^n}{n^k}\\ \mathrm{Li}'_k(x) &= \sum_{n=1}^{\infty} \frac{x^{n-1}}{n^{k-1}}\\ x\,\mathrm{Li}'_k(x) &= \sum_{n=1}^{\infty} \frac{x^n}{n^{k-1}}\\ &= \mathrm{Li}_{k-1}(x).\end{aligned}$$

The other form of the formula can be obtained by dividing by x and integrating on the above result. ■

As was previously mentioned for k, certain values of x result in $\mathrm{Li}_k(x)$ resembling other common functions. Here are two examples:

$$\begin{aligned}\mathrm{Li}_k(1) &= \sum_{n=1}^{\infty} \frac{1}{n^k} = \zeta(k)\\ \text{and } \mathrm{Li}_k(-1) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^k} = \eta(k),\end{aligned}$$

where $\zeta(k)$ is the Riemann zeta function and $\eta(k)$ is the Dirichlet eta function, also known as the alternating zeta function.

2. The Dilogarithm

Now, let us take a look at one of the specific cases of the polylogarithm function. While there are many of these (in fact, infinitely many), the dilogarithm has several interesting properties, from special values to functional equations. Let's look at some of these.

Definition 2.1. The *dilogarithm function*, also known as *Spence's function* (named after the early ninteenth century Scottish mathematician William Spence), is a special case of the polylogarithm for $k = 2$, given by

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}.$$

Proposition 2.2. *One of the aforementioned functional equations is a reflection formula, given by*

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \frac{\pi^2}{6} - \log(x)\log(1-x).$$

Proof. We begin with an integral representation of the dilogarithm function, given by

$$\mathrm{Li}_2(x) = -\int \frac{\log(1-x)}{x}\,dx.$$

By substituting $1-x$ in place of x, we get

$$\mathrm{Li}_2(1-x) = \int \frac{\log(x)}{1-x}\,dx.$$

Then, adding the two equations above and integrating gives us

$$\begin{aligned}\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) &= -\int \left(\frac{\log(1-x)}{x} - \frac{\log(x)}{1-x}\right) dx \\ &= -\log(x)\log(1-x) + C.\end{aligned}$$

Now, we solve for the constant of integration by plugging in $x = 0$.

$$C = \mathrm{Li}_2(1) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

Thus,

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \frac{\pi^2}{6} - \log(x)\log(1-x).$$

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Proposition 2.3 (Abel's Duplication Formula)**.** *Another important relation is the duplication formula, which obviously follows from Proposition 1.2 and states*

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(-x) = \tfrac{1}{2}\,\mathrm{Li}_2(x^2).$$

Proposition 2.4. *We have the following five-term functional equation, which can be expressed in many other forms as well:*

$$\mathrm{Li}_2\left(\frac{x}{1-x}\cdot\frac{y}{1-y}\right) = \mathrm{Li}_2\left(\frac{y}{1-x}\right) + \mathrm{Li}_2\left(\frac{x}{1-y}\right) - \mathrm{Li}_2(y) - \mathrm{Li}_2(x) - \log(1-y)\log(1-x).$$

Proof. If we take the integral representation of the dilogarithm

$$\mathrm{Li}_2(x) = -\int \frac{\log(1-x)}{x}\,dx.$$

and replace x with $\frac{a}{1-a}\cdot\frac{y}{1-y}$, treating a as a constant, we obtain the equation

$$\mathrm{Li}_2\left(\frac{a}{1-a}\cdot\frac{y}{1-y}\right) = -\int \left(\frac{1}{y} + \frac{1}{1-y}\right) \log\frac{1-a-y}{(1-a)(1-y)}\,dy.$$

Expanding the fractions and logarithm, we get

$$\begin{aligned}\operatorname{Li}_2\left(\frac{a}{1-a}\cdot\frac{y}{1-y}\right) = -\int \frac{1}{y}\log\left(1-\frac{y}{1-a}\right)dy + \int \frac{\log(1-y)}{y}\,dy \\ -\int \frac{1}{1-y}\log\left(1-\frac{a}{1-y}\right)dy + \int \frac{\log(1-a)}{1-y}\,dy.\end{aligned}$$

All of the integrals on the right hand side of the above equation can be written in terms of the logarithm and dilogarithm by applying the following three relations:

$$\begin{aligned}\int \frac{1}{y}\log\left(1-\frac{y}{1-a}\right)dy &= -\operatorname{Li}_2\left(\frac{y}{1-a}\right),\\ \int \frac{\log(1-y)}{y}\,dy &= -\operatorname{Li}_2(y),\\ \text{and}\quad \int \frac{1}{1-y}\log\left(1-\frac{a}{1-y}\right)dy &= -\operatorname{Li}_2\left(\frac{a}{1-y}\right).\end{aligned}$$

After making the appropriate substitutions and adding the constant of integration, the resulting equation is

$$\operatorname{Li}_2\left(\frac{a}{1-a}\cdot\frac{y}{1-y}\right) = \operatorname{Li}_2\left(\frac{y}{1-a}\right) + \operatorname{Li}_2\left(\frac{a}{1-y}\right) - \operatorname{Li}_2(y) - \log(1-a)\log(1-y) + C.$$

By plugging in $y = 0$, we see that $C = -\operatorname{Li}_2(a)$. Then, replacing a with x, we obtain the desired result:

$$\operatorname{Li}_2\left(\frac{x}{1-x}\cdot\frac{y}{1-y}\right) = \operatorname{Li}_2\left(\frac{y}{1-x}\right) + \operatorname{Li}_2\left(\frac{x}{1-y}\right) - \operatorname{Li}_2(y) - \operatorname{Li}_2(x) - \log(1-y)\log(1-x).$$

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A few other useful two-term functional equations arise from substitutions into the above five-term equation, including the following:

Proposition 2.5.

$$\operatorname{Li}_2(1-z) + \operatorname{Li}_2\left(1-\frac{1}{z}\right) = -\tfrac{1}{2}\log^2 z$$

Proof.

$$\operatorname{Li}_2\left(\frac{x}{1-x}\cdot\frac{y}{1-y}\right) = \operatorname{Li}_2\left(\frac{y}{1-x}\right) + \operatorname{Li}_2\left(\frac{x}{1-y}\right) - \operatorname{Li}_2(y) - \operatorname{Li}_2(x) - \log(1-y)\log(1-x).$$

Now, we make the substitution $x = y = 1-z$ and we get

$$\operatorname{Li}_2\left(\left(\frac{1-z}{z}\right)^2\right) = 2\operatorname{Li}_2\left(\frac{1-z}{z}\right) - 2\operatorname{Li}_2(1-z) - \log^2 z.$$

We can apply the square relationship on the left hand side of the above equation, resulting in the following:

$$2\operatorname{Li}_2\left(\frac{1-z}{z}\right) + 2\operatorname{Li}_2\left(\frac{z-1}{z}\right) = 2\operatorname{Li}_2\left(\frac{1-z}{z}\right) - 2\operatorname{Li}_2(1-z) - \log^2 z.$$

Moving all the dilogarithm terms to the left hand side and simplifying, we obtain the desired result:

$$\mathrm{Li}_2(1-z) + \mathrm{Li}_2\left(1 - \frac{1}{z}\right) = -\tfrac{1}{2}\log^2 z.$$

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Proposition 2.6 (Inversion Formula)**.** *Finally, we have the* inversion formula*, which relates the dilogarithm of an argument and the dilogarithm of its reciprocal.*

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1/z) = -\tfrac{\pi^2}{6} - \tfrac{1}{2}\log^2 z.$$

Since we have relations for dilogarithms of the quantities

$$x,\ \frac{1}{x},\ 1-x,\ \frac{1}{1-x},\ \frac{x-1}{x},\ \text{and}\ \frac{x}{x-1},$$

we are able to reduce any real argument down to $|x| < \frac{1}{2}$. This allows us to compute particular values of the dilogarithm efficiently.

Notably, there are only eleven known values for x for which both x and $\mathrm{Li}_2(x)$ can be expressed in closed form. Most of these can be found using the identities listed above. These are

$$\begin{aligned}
\mathrm{Li}_2(0) &= 0,\\
\mathrm{Li}_2(1) &= \frac{\pi^2}{6},\\
\mathrm{Li}_2(-1) &= -\frac{\pi^2}{12},\\
\mathrm{Li}_2\left(\frac{1}{2}\right) &= \frac{\pi^2}{12} - \frac{1}{2}\log^2(2),\\
\mathrm{Li}_2(2) &= \frac{\pi^2}{4} - \pi i \log 2,\\
\mathrm{Li}_2(\phi^{-2}) &= \frac{\pi^2}{15} - \log^2(\phi),\\
\mathrm{Li}_2(\phi^{-1}) &= \frac{\pi^2}{10} - \log^2(\phi),\\
\mathrm{Li}_2(-\phi^{-1}) &= -\frac{\pi^2}{15} + \log^2(\phi),\\
\mathrm{Li}_2(\phi) &= \frac{11\pi^2}{15} + \frac{1}{2}\log^2(-\phi^{-1}),\\
\mathrm{Li}_2(\phi^2) &= -\frac{11\pi^2}{15} - \log^2(-\phi),\\
\text{and}\ \mathrm{Li}_2(-\phi) &= -\frac{\pi^2}{10} + \log^2(\phi),
\end{aligned}$$

where ϕ is the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$.

3. Multiple Polylogarithms

There is also a multiple polylogarithm function, which involves multiple orders as well as multiple arguments.

Definition 3.1. The *multiple polylogarithm* function of depth k is defined as

$$\mathrm{Li}_{s_1,\ldots,s_k}(x_1,\ldots,x_k)=\sum_{n_1>\cdots>n_k\geq 1}\frac{x_1^{n_1}\cdots x_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}}.$$

Sometimes, the multiple polylogarithm is used on only one variable. In such situations, it is defined as

$$\mathrm{Li}_{s_1,\ldots,s_k}(x)=\sum_{n_1>\cdots>n_k\geq 1}\frac{x^{n_1}}{n_1^{s_1}\cdots n_k^{s_k}}.$$

The multiple polylogarithm in one variable has several properties that are similar to the classical single polylogarithm. These include the following recurrence formulas:

$$x\frac{d}{dx}\mathrm{Li}_{s_1,\ldots,s_k}(x)=\mathrm{Li}_{s_1-1,\ldots,s_k}(x)$$

$$\text{and } (1-x)\frac{d}{dx}\mathrm{Li}_{1,s_2,\ldots,s_k}(x)=\mathrm{Li}_{s_2,\ldots,s_k}(x).$$

Along with the initial condition

$$\mathrm{Li}_{s_1,\ldots,s_k}(0)=0,$$

the above differential equations uniquely determine the multiple polylogarithm for all valid $s_1,\ldots,s_k$.

4. Multiple Zeta Values

Definition 4.1. The *multiple zeta function* is a generalization of the Riemann zeta function to multiple variables and is defined as

$$\zeta(s_1,\ldots,s_k)=\sum_{n_1>\cdots>n_k\geq 1}\frac{1}{n_1^{s_1}\cdots n_k^{s_k}}.$$

We shall begin with some relevant vocabulary. When $s_1,s_2,\ldots$ are positive integers with $s_1>1$, the sums are referred to as multiple zeta values (MZVs). The quantity $s_1+s_2+\cdots+s_k$ is the weight of an MZV and k is its length.

Note that, similar to the single zeta function and single polylogarithm, the following is true:

$$\mathrm{Li}_{s_1,\ldots,s_k}(1)=\zeta(s_1,\ldots,s_k).$$

Proposition 4.2. *One equation that can be used to translate some instances of multiple zeta values into single zeta values is*

$$2\zeta(s,s)=\zeta(s)^2-\zeta(2s).$$

Proof. The proof is shown by eliminating the condition $m < n$, splitting the series, and expressing the terms with forms of the single zeta function.

$$\begin{aligned}\zeta(s,s) &= \sum_{0<m<n} \frac{1}{m^s n^s} \\ &= \frac{1}{2} \sum_{\substack{m,n>0 \\ m\neq n}} \frac{1}{m^s n^s} \\ &= \frac{1}{2}\left(\sum_{m,n>0} \frac{1}{m^s n^s} - \sum_{n>0} \frac{1}{n^{2s}} \right) \\ &= \tfrac{1}{2}(\zeta(s)^2 - \zeta(2s))\end{aligned}$$

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