Apery's Theorem

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June 2019

1 Apery's Theorem

This theorem states that $\zeta 3$ is irrational.

2 Proof of Apery's Theorem

We first prove that $\zeta 2$ equals $\frac{(\pi)^2}{6}$.

$$\zeta(s) = \prod_{p=prime} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots\right).$$

The stuff inside the parentheses is a geometric series, with sum $\left(1 - \frac{1}{p^s}\right)^{-1}$. This gives the desired result. Using this, we can get new identities about Dirichlet series not coming from Dirichlet convolutions. Let's start with the Riemann zeta function and then take the logarithmic derivative with respect to s. That means that we take a logarithm and then take the derivative $\frac{d}{ds}\log\zeta(s)$. When we apply this to the Riemann zeta function, we get

$$\begin{aligned}
\zeta'(s) &= \frac{d}{ds}\log\zeta(s) \\
&= \frac{d}{ds}\log\left(\prod_{p=\text{prime}}\left(1 - \frac{1}{p^s}\right)^{-1}\right) \\
&= -\frac{d}{ds}\sum_{p=\text{prime}}\log\left(1 - \frac{1}{p^s}\right) \\
&= -\sum_{p=\text{prime}}\left(\frac{\log(p)}{p^s}\right)\left(1 - \frac{1}{p^s}\right)^{-1} \\
&= -\sum_{p=\text{prime}}\log(p)\sum_{m=1}^{\infty}\frac{1}{p^{ms}} \\
&= -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}.
\end{aligned}$$

We have

$$\begin{aligned}
\zeta(2s) = &\left(\prod_{p=\text{ prime}}\left(1 - \frac{1}{p^s}\right)^{-1}\right)\left(\prod_{p=\text{ prime}}\left(1 - \frac{1}{p^{2s}}\right)\right) \\
&= \prod_{p=\text{prime}}\frac{1 - p^{-2s}}{1 - p^{-s}} \\
&= \prod_{p=\text{prime}}\left(1 + \frac{1}{p^s}\right).
\end{aligned}$$

In other words, this is the Dirichlet Series for the multiplicative arithmetic function that takes on the value 1 at all primes and 0 at all higher prime powers,but that's just $|\mu|$. This is an integral involving a floor function, we should break it up into a sum over all possible values of the floor so it will be easier to compute. Thus we have

$$\begin{aligned}\int_1^\infty \frac{\lfloor x\rfloor}{x^{s+1}}\,dx &= \sum_{n=1}^\infty \int_n^{n+1} \frac{\lfloor x\rfloor}{x^{s+1}}\,dx \\ &= \sum_{n=1}^\infty \int_n^{n+1} \frac{n}{x^{s+1}}\,dx \\ &= \sum_{n=1}^\infty \left[\frac{n}{(-s)x^s}\right]_n^{n+1} \\ &= -\frac{1}{s}\sum_{n=1}^\infty \left(\frac{n}{(n+1)^s} - \frac{n}{n^s}\right) \\ &= -\frac{1}{s}\left[\frac{1}{2^s} - \frac{1}{1^s} + \frac{2}{3^s} - \frac{2}{2^s} + \frac{3}{4^s} - \frac{3}{3^s} + \frac{4}{5^s} - \frac{4}{4^s} + \cdots\right] \\ &= -\frac{1}{s}\left[-\frac{1}{1^s} - \frac{1}{2^s} - \frac{1}{3^s} - \cdots\right] \\ &= \frac{\zeta(s)}{s}.\end{aligned}$$

The result follows upon multiplication by s. We can let $s = 3$ since we are trying to figure out $\zeta 3$. Thus we have that $\zeta(3) = 3$ times the integral. This will lead to us showing that the integral is irrational.

3 Apery's Original Proof

Apery's original proof was based on the well known irrationality criterion from Peter Gustav Lejeune Dirichlet, which states that a number ξ is irrational if there are infinitely many co-prime integers p and q such that
$\left|\xi - \frac{p}{q}\right| < \frac{c}{q^{1+\delta}}$ for some fixed c,$\delta > 0$.
The starting point for Apery's proof was the series representation of $\zeta(3)$ as

$$\zeta(3) = \frac{5}{2}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^3 * 2nCn}.$$

Roughly speaking, Apery then defined a sequence c_n,k which converges to $\zeta 3$ about as fast as the above series, specifically

$$c_{n,k} = \sum_{m=1}^n \frac{1}{m^3} + \sum_{m=1}^k \frac{(-1)^{m-1}}{2m^3 * nCm * n + mCm}.$$

He then defined two more sequences a_n and b_n that, roughly, have the quotient c_n,k. These sequences were

$$a_n = \sum_{k=0}^n c_{n,k}(nCk)^2 * ((n+k)Ck)^2$$

$$b_n = \sum_{k=0}^n (nCk)^2 * ((n+k)Ck)^2$$

The sequence $\frac{a_n}{b_n}$ converges to $\zeta 3$ fast enough to apply the criterion, but unfortunately an is not an integer after n=2. Nevertheless, Apery then showed that even after multiplying a_n and b_n by a suitable integer to fix the problem. The convergence is still fast enough to guarantee irrationality.

4 Proof by Fritz Beuker

An alternative, shorter proof was proposed by Beuker. He interpreted Apery's approximations by integrals. From the definition of Legendre polynomials $L_n(x)$ as an nth derivative and the fact that

$$\int_0^1 \int_0^1 \int_0^1 \frac{L_n(x)L_n(y)}{1 - xz + xyz} dxdydz = (A_n + B_n\zeta 3)l_n^3$$

for integers A_n and B_n, Beukers derived by integration by parts that the integral above is nonzero and asymptotically small. The irrationality of $\zeta 3$ follows by the same final arguments as in Apery's proof.

5 Another proof

Suppose the contradiction to Apery's theorem, that $\zeta 3 = p/$q, where p and q are positive integers. Then, using a trivial bound $D_n < 3^n$, we deduce that, for each n = 0, 1, 2, . . ., the integer $qD_n^3 F_n = D_n^3 u_n p - D_n^3 v_n q$ satisfies the estimate $0 < qD_n^3 F_n < 20q(n+1)^4 3^3 n(\sqrt{2}-1)^{4n}$ that is not possible since $3^3(\sqrt{2}-1)^4 = 0.7948... <$ 1 and the right-hand side of the equation above is less than 1 for a sufficiently large integer n. This contradiction completes the proof of the theorem.

Mathematicians later on proved that zeta of all odd numbers are irrational. Zeta of all negative even numbers are equal to 0, while sets of all negative odd number are rational.

6 Zeta of negative numbers

$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$
Zeros occur at the negative even integers:
$\zeta(-2n) = 0$
$\zeta(-1) = -\frac{1}{12}$
$\zeta(-3) = \frac{1}{120}$
$\zeta(-5) = -\frac{1}{252}$
$\zeta(-7) = \frac{1}{240}$
$\zeta(-9) = -\frac{1}{132}$
$\zeta(-11) = \frac{691}{32760}$
$\zeta(-13) = -\frac{1}{12}$
These are the first few negative odd numbers, we can see that all of them are rational unlike their positive counterparts and negative even numbers equal zero unlike their positive counterparts.