ORIGAMI NUMBERS

WILLIAM LEE

Abstract. We examine the field of numbers that can be constructed using folds. We prove results similar to those proved in the classical ruler compass constructions.

1. Introduction

Origami constructions are an lesser known variant of straight-edge compass constructions. When considering origami constructions, our paper is a plane, on which we are given two points, and we can subsequently fold lines which align given points or lines on the plane. Furthermore, we can fold the point that is the intersection of two lines, but not necessarily every point on a line. Note that each of these are single folds, meaning that after each fold, the paper is immediately unfolded, leaving a crease. It turns out that these folds are in fact more powerful than ordinary ruler compass constructions. We will first discuss the allowed folds, and define origami numbers and points. Then we will characterize all of the origami numbers, similar to the characterization of constructible numbers. Finally, we will revisit the problems of trisecting and angle, doubling a cube, and foldable regular polygons with origami numbers.

2. Origami Numbers

Before we can give a definition of origami numbers, we must describe what unique folds are possible in each single fold construction. We use the following set of Axioms, known as the Huzita-Hatori Axioms.

(1) Given two points p_1, p_2, we can fold the line ℓ that connects them.
(2) Given two points p_1, p_2, we can fold the line reflecting p_1 onto p_2. (perpendicular bisector)
(3) Giver two lines ℓ_1, ℓ_2, we can fold the line placing ℓ_1 onto ℓ_2. (angle bisector)
(4) Given Point p_1 and line ℓ, we can fold a line perpendicular to ℓ through p_1 (perpendicular)
(5) Given two points p_1, p_2, and a line ℓ, we can fold the line through p_1 reflecting p_2 onto ℓ.

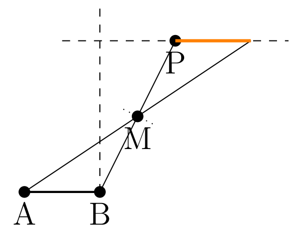

FIGURE 1. Parallels

(6) Given two points p_1, p_2 , and two lines ℓ_1, ℓ_2, we can fold the line which reflects p_1 onto ℓ_1, and p_2 onto ℓ_2.

(7) Given a point p_1, and two lines ℓ_1, ℓ_2, we can fold a line perpendicular to ℓ_2 that reflecting p_1 onto ℓ_1

Each of the first 4 folds are straight forward to perform. For Axiom (5), we take p_1, and slide the paper until p_2 coincides with ℓ. For (6), we place p_1 onto ℓ_1, and slide it along ℓ_1 until p_2 coincides with ℓ_2. For (7), we place ℓ_2 onto itself, and slide until p_1 coincides with ℓ_1.

These Axioms were proved to be the complete set of axioms for single fold origami folds.

Now, we can define an origami number.

Definition. A length r is origami foldable if and only if starting from two points of unit distance apart, we perform a sequence of single folds resulting in 2 points r units apart. A real number r is an origami number if and only if $|r|$ is and origami foldable length.

Definition. A point (a, b) in the plane is a origami point if and only if one can fold it in a finite amount of single folds given the points $(0, 0), (1, 0)$.

3. CONSTRUCTIONS

In this section, we find a classification of the origami numbers. First, we show the following useful property:

Lemma 1. *Given a segment AB, and a point P, we can fold a segment parallel to AB through P, with length equal to AB.*

Remark 1. In [1], Alperin shows that we can achieve this with only operations (1) – (2), but it is slightly longer.

This construction is shown in figure 1. A result of this is the another useful property:

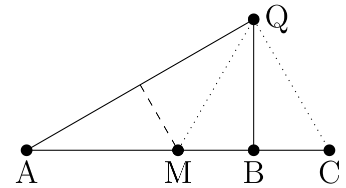

FIGURE 2. Folding $\sqrt{a}$

Lemma 2. *Given a segment AB with length r, and a point P on a line ℓ, we can fold the point P' on ℓ with the length of PP' is r.*

Proof. By lemma 1, we can fold the segment PQ of length r that starts at P, parallel to AB. Now bisect the angle given by segment PQ and line ℓ. Finally, P' is the intersection of the perpendicular from Q to this bisector and ℓ. □

Corollary 3. (a, b) is a origami point if and only if a, b are origami numbers.

Proof. First note that we can fold the x and y axis. Now, if we have origami point (a, b), we can fold the perpendiculars from the point to the x and y axis to get lengths a, b. Now, if a, b are origami numbers, just use lemma 2 to translate it to the origin on the axis, and draw perpendiculars. □

With these, we can now show that the Origami numbers form a field, and contains the constructible numbers.

Theorem 4. *The set of Origami numbers is closed under addition, subtraction, multiplication, division, and square roots.*

Proof. Note that with our Lemmas, and operations (1)-(4), we can do everything that is required for the normal straight edge compass additions, subtractions, multiplication, and division. The square roots takes a bit more work, as we cannot fold circles. Nonetheless, we can still do it with the help of operation (5), as follows. See Figure 2 for reference. fold three collinear points A, B, C with $AB = a$, $BC = 1$. Also, fold the midpoint M of $\overline{AC}$. Now, by property (5), fold the line through M that reflects A onto the perpendicular through B. Let Q be the intersection between the perpendicular from A to this line and the perpendicular from B to $\overline{AC}$. $\overline{QB}$ is $\sqrt{a}$ □

Remark 2. It is also possible to show that $\sqrt{a}$ is foldable algebraically, noting that (5) corresponds to drawing tangents to a parabola. [1].

Thus, the constructible numbers is a subfield of the origami numbers. It is worth noting that each of for all of this, we have only used operations $(1) - (5)$. In fact, Axioms $(1) - (5)$ are exactly the constructible numbers.

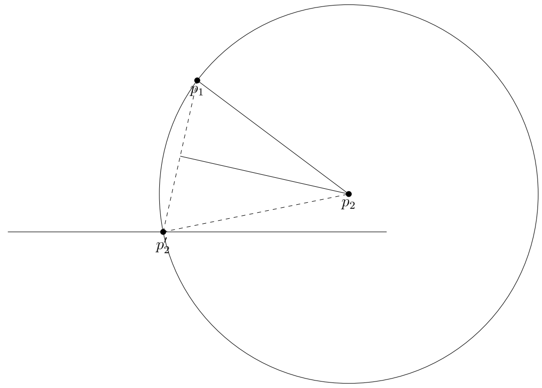

FIGURE 3. Ruler compass (5)

Theorem 5. *Operations* (1) – (5) *determine exactly the constructible numbers.*

Proof. Theorem 4 shows that every constructible number can be folded with axioms (1) – (5).

Thus, it suffices to show that each of (1) – (5) can be achieved through ruler and compass. Axioms (1) – (4) are easy. Now, we consider (5). See Figure 3. Consider points p_1, p_2, and line ℓ. If there exists a line ℓ_1 such that ℓ_1 passes through p_1, and reflects p_2 onto ℓ, let p_2' be the reflection of p_2 over ℓ_1. Now, p_1 is equidistant from p_2, and p_2', thus the circle centered at p_1 passing through p_2 passes through p_2', so we can construct p_2'. Now, to construct ℓ_1, Just take the perpendicular bisector of $\overline{p_2 p_2'}$. □

Theorem 6. *The foldable numbers are closed under cube roots.*

Proof. We use the method due to Beloch described in [4]. See Figure 4. We use coordinates. For a origami number, a, we wish to fold $\sqrt[3]{a}$. Let $P = (0, -1)$, $Q = (-a, 0)$. Using (6), consider the fold placing P onto $y = 1$, and Q on $x = a$. Let the crease created by this fold be ℓ, and let $P \mapsto P'$, $Q \mapsto Q'$.Consider Y, the intersection of PP' and ℓ. Y is equidistant to P and P', and thus lies on the x-axis. Similarly, X, the intersection of QQ' and ℓ lies on the Y-axis. Now, consider triangles QXO, XOY, YOP. They are all similar, thus $\frac{QO}{XO} = \frac{XO}{YO} = \frac{YO}{OP}$. Substituting the values $QO = a, OP = 1$, and solving for YO in terms of a, we obtain, $YO = \sqrt[3]{a}$, thus $\sqrt[3]{a}$ is indeed foldable. □

Corollary 7. Any cubic in the field of origami numbers has real roots in the origami numbers.

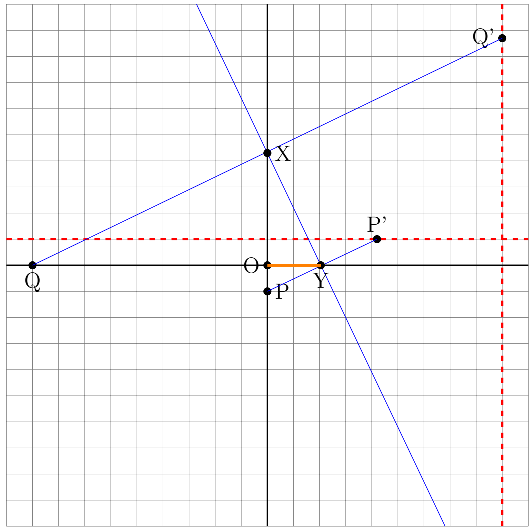

FIGURE 4. Folding $\sqrt[3]{a}$

Proof. This follows from the cubic formula, it only requires cube roots and square roots. □

Thus, we have shown that the set of origami numbers form a field closed under square roots and cube roots. It turns out, this completely characterizes the origami numbers- it is the smallest subfield of $\mathbb{R}$ closed under square roots and cube roots. We show this in the following theorem.

Theorem 8. *$r \in \mathbb{R}$ is origami foldable if and only if there is a finite sequence of fields $\mathbb{Q} = F_0 \subset F_1 \subset \cdots \subset F_n \subset \mathbb{R}$ such that $r \in F_n$, and $[F_i : F_{i-1}] = 2$ or 3 for each $1 \leq i \leq n$*

Proof. If there exists such a sequence of fields, then we can certainly fold r, as from our above work, the origami numbers are closed under square and cuberoots. So, it remains to prove the converse.

We only give an outline of the proof, leaving out much of the computations. Interested readers may see [1], or [2]. Suppose at one point in the folding, every folded point has coordinates in a field F. We must show that each new point lies in a series of quadratic or cubic extensions over F. Note that every new point is an intersection of two lines, and if the lines have coefficients in a quadratic or cubic extension of F, then their intersection does also. Thus, we just need to show that every fold created from F has coefficients in a quadratic or cubic extension of F. In what follows, we assume that all current lines are of this form, to show that any newly folded line must be of this form also. For operations $(1) - (5)$, it is clear, as they only involve quadratic extensions. Furthermore, one can find that (6) only involves a cubic extension, and (7) only involves a quadratic one. Thus, in every case, we have a equation with coefficients in a quadratic or cubic extension of F, and at the beginning, we also do, so we are done. □

Remark 3. To show that (7) is in a quadratic extension, one can show a construction of (7) with ruler and compass. Showing that (6) lies in a cubic extension is a bit harder, but one can do so by showing that (6) is equivalent to finding the simultaneous tangents between two parabolas with foci at p_1, p_2 and directrixes at ℓ_1, ℓ_2.

Remark 4. This shows that origami constructions are equivalent to conic constructions, as in [4].

4. Greek Problems

From our above discussion of origami numbers, it is clear that through paper folding, we can double a cube, and trisect an arbitrary angle, as they only involve cubic equations. In fact, doubling the cube follows directly from Theorem 6. As for the problem of squaring the circle, it is still unsolvable, as π is transcendental.

5. Regular Polygons

Now, we turn towards the problem of which regular polygons are foldable. We use a similar method as the proof of ruler compass constructible polygons.

Lemma 9. *If K/F is Galios with $[K : F] = 2^\alpha 3^\beta$, then there exists a sequence of fields $F = F_0 \subset F_1 \subset \cdots \subset F_n = K$ such that $[F_i : F_{i-1}] = 2$ or 3 for all i.*

Proof. We follow the proof in [4]. Let $G = \mathrm{Gal}(K/F)$. By Burnside's p-q theorem, G is solvable, thus there exists a composition series

$$G = G_0 \rhd G_1 \rhd \cdots \rhd G_n = \{e\}$$

, such that each of the quotients are abelian. Now, I claim that each of the quotients must be of order a prime, specifically, 2 or 3. To see this, note that each of the Quotients are simple abelian groups, and the only simple abelian groups are of prime order (If the order were not prime, then by cauchy's theorem, there exists a subgroup, but as it is abelian, every subgroup is normal to it, and thus cannot be simple).

Now, to find our desired sequence of Fields, we take the sequence $F = F_0 \subset F_1 \subset \cdots \subset F_n = K$, where each $F_j = K^{G_j}$. I claim that each $[F_i : F_{i-1}] = 2$ or 3. We have $[F_i : F_{i-1}] = [K^{G_i} : K^{G_{i-1}}] = |\,\mathrm{Gal}(K^{G_i}/K^{G_{i-1}})| = |\,\mathrm{Gal}(K/K^{G_{i-1}})/G_i| = |G_{i-1}/G_i| = 2$ or 3. □

Now, we are ready to prove our theorem.

Theorem 10. *A regular n-gon is origami constructible if and only if $n = 2^a 3^b p_1 p_2 \cdots p_r$, for some $r \in \mathbb{N}$ where each p_i are distinct primes of the form $2^c 3^d + 1$*

Proof. We first show that is a regular n-gon is origami foldable, then $n = 2^a 3^b p_1 p_2 \cdots p_r$. Without loss of generality, let the n-gon be centered at the origin, with one vertex at $(1, 0)$. If we can fold this n-gon, then we can fold the next vertex, at $(\cos \frac{2\pi}{n}, \sin \frac{2\pi}{n})$, so $\cos \frac{2\pi}{n}$ is origami. Thus, $[\mathbb{Q}(\cos \frac{2\pi i}{n}) : \mathbb{Q}] = 2^\alpha 3^\beta$, for integers α, β.

Now, I claim $\mathbb{Q}(\zeta_n) = \mathbb{Q}(\cos \frac{2\pi}{n}, i \sin \frac{2\pi}{n})$. Note that $\zeta_n = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$, thus $\mathbb{Q}(\zeta_n) \subseteq \mathbb{Q}(\cos \frac{2\pi}{n}, i \sin \frac{2\pi}{n})$. Furthermore, $\cos \frac{2\pi}{n} = \frac{\zeta_n + \zeta_n^{-1}}{2}$, and $i \sin \frac{2\pi}{n} = \frac{\zeta_n - \zeta_n^{-1}}{2}$, thus $\mathbb{Q}(\cos \frac{2\pi}{n}, i \sin \frac{2\pi}{n}) \subseteq \mathbb{Q}(\zeta_n)$, so they are equal, as we wanted.

Now, $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = [\mathbb{Q}(\cos \frac{2\pi}{n}, i \sin \frac{2\pi}{n}), \mathbb{Q})] = [\mathbb{Q}(\cos \frac{2\pi}{n}, i \sin \frac{2\pi}{n}) : \mathbb{Q}(\cos \frac{2\pi}{n}][\mathbb{Q}(\cos \frac{2\pi}{n} : \mathbb{Q}]$. Also, $[\mathbb{Q}(\cos \frac{2\pi}{n}, i \sin \frac{2\pi}{n}) : \mathbb{Q}(\cos \frac{2\pi}{n}] = 2$, as $i \sin \frac{2\pi}{n}$ is a root of $x^2 - \cos^2 \frac{2\pi}{n} + 1$. Thus $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = 2^{\alpha+1} 3^\beta$.

But,

$$[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \phi(n) = p_1^{e_1 - 1} p_2^{e_2 - 1} \cdots p_r^{e_r - 1} (p_1 - 1) \cdots (p_r - 1)$$

where $p_1^{e_1} \cdots p_r^{e_r}$ is the prime factorization of n. So, $p_1^{e_1 - 1} \cdots p_2^{e_2 - 1} (p_1 - 1) \cdots (p_r - 1) = 2^{\alpha+1} 3^\beta$, which is enough to imply that n is indeed of our desired form.

Now, we must show the reverse direction, if $n = 2^a 3^b p_1 p_2 \cdots p_r$, then we can fold a n-gon. Note that if we can fold $\cos \frac{2\pi}{n}$, then we can fold the regular polygon centered at the origin with vertex at $(1, 0)$. Indeed, we can then fold the point $(\cos \frac{2\pi}{n}, \sin \frac{2\pi}{n})$, as $\sin \frac{2\pi}{n} = \sqrt{1 - \cos^2 \frac{2\pi}{n}}$ is foldable. Then, after that, we can symmetrically fold the rest of the coordinates of the polygon. Thus, it suffices to show that we can fold $\cos \frac{2\pi}{n}$. Note that $\mathbb{Q}(\cos \frac{2\pi}{n})/\mathbb{Q}$ is Galios. To see this, note that $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ is abelian, so $A = \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}(\cos \frac{2\pi}{n})) \triangleleft \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, thus $\mathbb{Q}(\zeta_n)^A = \mathbb{Q}(\cos \frac{2\pi}{n})$ is a normal extension of $\mathbb{Q}$, and is thus Galios as separability carries over from being a subfield of $\mathbb{Q}(\zeta_n)$.

Now, note that $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \phi(n) = 2^\alpha 3^\beta$ for some α, β in the integers, and as with the preceeding discussion while proving the other direction, $[\mathbb{Q}(\cos \frac{2\pi}{n}) : \mathbb{Q}] = 2^{\alpha-1} 3^\beta$. Now, as $\mathbb{Q}(\cos \frac{2\pi}{n})/\mathbb{Q}$ is Galios, Lemma 9 implies the result.

□

References

[1] http://nyjm.albany.edu/j/2000/6-8.pdf

[2] https://getd.libs.uga.edu/pdfs/lee_hwa-young_201712_ma.pdf

[3] http://origametry.net/papers/amer.math.monthly.118.04.307-hull.pdf

[4] https://www.researchgate.net/publication/227141144_On_points_constructible_from_conics