Category theory

William Ren
Euler Circle
Palo Alto CA 94303, USA
walumren@gmail.com

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Abstract

In this paper, we are going to go through Objects and their various morphisms. Additionally we will briefly cover functors and natural transformations. This subject is about generalizing groups, manifolds, metric spaces, topological spaces, and even Euclidean geometry. For example in Euclidean geometry when we talk about mapping we are talking about rigid motions. $f : \mathbb{R}^2 \to \mathbb{R}^2$ that maps $d < x, y > tod < Tx, Ty >$. Additonally for Abstract Algebra where we look at homomorphsisms $f :< G, \bullet > \to < H, \times >$ Such that $f(g \bullet h) = f(g) \times f(h)$. What these all have in common is that they all are structure preserving maps.

We then see how a category generalizes all of these structures from these seemingly very different topics, all these ideas can be unified by a category, where the idea is think only about the objects and the arrows that are between them, without focusing too hard about what the objects are. Their is a very convinent way of representing these ideas by way of a commutative diagram. We will towards the end cover what is called a functor.

1 Mappings

We again just like the summary and discuss the motivation for the general definition of a Category. To introduce categories we are going to look at objects and their maps, to be more specific their isomorphisms. Additonally for Abstract Algebra where we look at homomorphsisms $f :< G, \bullet > \to < H, \times >$ which preserves this property $f(g \bullet h) = f(g) \times f(h)$.

Two groups G and H are isomorphic if there was a bijective homomorphism: or equivalently, we wanted homomorphisms $\phi : G \to H$ and $\psi : H \to G$ which were mutual inverses, meaning $\phi \circ \psi =_H$ and $\psi \circ \phi =_G$.

For example in Euclidean geometry when we talk about mapping we are talking about rigid motions. $f : \mathbb{R}^2 \to \mathbb{R}^2$ that maps $d < x, y > tod < Tx, Ty >$.Which is a mapping that preserves distance

Two vector spaces V and W are isomorphic if there is a bijection $T : V \to W$ which is a linear map.

Definition 1.1. Now we will introduce what a category actually is and it's definition

A category consists of objects which is usually denoted as $(\mathbb{A})$

For any two objects $A_1, A_2 \in (\mathbb{A})$, there exists arrows between them (also referred to as morphisms or maps) . We'll denote the set arrows referring to A_1, A_2 as $\mathrm{Hom}_{\mathbb{A}}(A_1, A_2)$.

For any $A_1, A_2, A_3 \in (\mathbb{A})$,if $f : A_1 \to A_2$ is an arrow and $g : A_2 \to A_3$ is an arrow, we can compose these arrows to get an arrow $g \circ f : A_1 \to A_3$.

We can represent this in a commutative diagram

$$\begin{tikzcd} A \arrow[r, tail] \arrow[dr] & B \arrow[d] \\ & C \end{tikzcd}$$

where $h = g \circ f$. The composition operation $\circ$ is part of the data of $\mathbb{A}$ implies it must be associative. In the diagram above we say that h factors through A_2. We have the idendity $\in (\mathbb{A})$ has a special identity arrow $_A$ Which does what it says it does.

Let us now use the definitions to describe to the examples that were given earlier. Examples of categories (some from earlier) There is a category of groups Grp. The data is

> The objects of Grp are the groups. The arrows of Grp are the homomorphisms between these groups. The composition $\circ$ in Grp is function composition.

In the same way it is clear to see a category $CRing$ of (commutative) rings. As staated before we have Top of topological spaces, whose arrows are the continuous maps. We have a category Set of sets, where the arrows are *any* maps.

An arrow $A_1 f A_2$ is an isomorphism if and only if there exists some $A_2 g A_1$ such that $f \circ g =_{A_2}$ and $g \circ f =_{A_1}$. In that case we say A_1 and A_2 are isomorphic, which implies $A_1 \cong A$

This is pretty cool, because we are now able to define whether two structures are the same based on the functions between those objects

One can learn about objects by the functions between them. Category theory takes this to the extreme by *only* looking at arrows, and ignoring what the objects themselves are.

But there are some trickier interesting examples of categories. [Posets are categories] Let $\mathcal{X}$ be a partially ordered set. We will X for it as follows:

> The objects of X are going to be the elements of $\mathcal{X}$. The arrows of X are defined as follows:For every object $x \in X$, we add an identity arrow $_p$, and For any pair of distinct objects $x \leq q$, we add a single arrow $x \to q$.

> Also there are no other arrows. There's only one way to do the composition.

Essentially this shows that arrows of a category can seem quite arbitrary and can be totally different from functions

Which is why we have the term "concrete category"which refers to the "structure-preserving maps between sets", that we talked about in the paper that motivate this topic like Grp, Top, or $CRing$.

This is another example. [Important: groups are one-Object categories] A group G can be interpreted as a category $\mathcal{G}$ with one object $*$, all of whose arrows are isomorphisms. A picture of this

As Tom Leninster says:

> The first time you meet the idea that a group is a kind of category, it's tempting to dismiss it as a coincidence or a trick. It's not: there's real content. To see this, suppose your education had been shuffled and you took a course on category theory before ever learning what a group was. Someone comes to you and says:
>
> "There are these structures called 'groups', and the idea is this: a group is what you get when you collect together all the symmetries of a given thing."
>
> "What do you mean by a 'symmetry'?" you ask.
>
> "Well, a symmetry of an object X is a way of transforming X or mapping X into itself, in an invertible way."
>
> "Oh," you reply, "that's a special case of an idea I've met before. A category is the structure formed by *lots* of objects and mappings between them – not necessarily invertible. A group's just the very special case where you've only got one object, and all the maps happen to be invertible."

Finally, here are some examples of categories you can make from other categories. [Deriving categories]

> [(a)] Given a category $\mathbb{A}$, we can construct the opposite category $\mathbb{A}$, which is the same as $\mathbb{A}$ but with all arrows reversed. Given categories $\mathbb{A}$ and , we can construct the product category $\mathbb{A}\times$ as follows: the objects are pairs (A, B) for $A \in \mathbb{A}$ and $B \in$, and the arrows from (A_1, B_1) to (A_2, B_2) are pairs
>
> $$(A_1 f A_2, B_1 g B_2) \, .$$

2 Special objects in categories

Set has initial object $\varnothing$ and final object $\{*\}$. An element of S corresponds to a map $\{*\} \to S$. Certain objects in categories have unique properties Here are a couple examples. [Initial object] An initial object of $\mathbb{A}$ is an object $A_{\text{init}} \in \mathbb{A}$ such that for any $A \in \mathbb{A}$ (possibly $A = A_{\text{init}}$), there is specifically one arrow from A_{init} to A. For example,

[(a)] The initial object of Set is the empty set $\varnothing$. The initial object of Grp is the trivial group $\{1\}$. The initial object of $CRing$ is the ring $\mathbb{Z}$ (recall that ring homomorphisms $R \to S$ map 1_R to 1_S). The initial object of Top is the empty space. The initial object of a partially ordered set is of its smallest element, if one exists.

We will refer to "the" initial object of a category, since:

In mathematics, we usually neither know nor care if two objects are actually equal or whether they are isomorphic. For example, there are many competing ways to define $\mathbb{R}$, but we refer to it as real numbers.

When we define these categorical notions, we check if they are unique up to to isomorphism.

One can take the "dual" notion, a terminal object. [Terminal object] A terminal object of $\mathbb{A}$ is an object $A_{\text{final}} \in \mathbb{A}$ such that for any $A \in \mathbb{A}$ (possibly $A = A_{\text{final}}$), there is exactly one arrow from A to A_{final}. For example,

[(a)] The terminal object of Set is the singleton set $\{*\}$. These are all isomorphic The terminal object of Grp is the trivial group $\{1\}$. The terminal object of $CRing$ is the zero ring 0. (Recall that ring homomorphisms $R \to S$ must map 1_R to 1_S). The terminal object of Top is the single-point space. The terminal object of a partially ordered set is its maximal element, if one exists.

Again, terminal objects are unique up to isomorphism. However, we will show how the notion of duality to give a short proof. Verify that terminal objects of $\mathbb{A}$ are equivalent to initial objects of $\mathbb{A}$. Thus terminal objects of $\mathbb{A}$ are unique up to isomorphism. In general, one can consider in this way the dual of *any* categorical notion: properties of $\mathbb{A}$ can all be translated to dual properties of $\mathbb{A}$ by adding the prefix "co" in front.

One last neat construction for this section : suppose we're working in a concrete category, meaning that the objects are sets with additional structure. Here is some ways to think about elements of these sets, even though it is category theory:

In Set, arrows from $\{*\}$ to S correspond to elements of S. In Top, arrows from $\{*\}$ to X correspond to points of X. In Grp, arrows from $\mathbb{Z}$ to G correspond to elements of G. In $CRing$, arrows from $\mathbb{Z}[x]$ to R correspond to elements of R.

and so on. So in a lot concrete categories, you can think of elements as functions from special sets to the set in question. In each of these cases we call the object in question a free object.

3 Binary products

The universal property is how you would describe objects in terms maps that defines the object such that it is unique to isomorphism

Here is an example of how this works. Suppose I'm in a category – let's say Set for now. I have two sets X and Y, and I want to construct the Cartesian product $X \times Y$ as we know it. In category theory we talk about maps without referring to the sets themselves

Well, let's think about maps into $X \times Y$. The key observation is that A function $AfX \times Y$ amounts to a pair of functions (AgX, AhY). Put another way, there is a natural projection map $X \times YX$ and $X \times YY$:

This is done because category theory is about the arrows. Now how do I add A to this diagram? The point is that there is a bijection between functions $AfX \times Y$ and pairs (g, h) of functions. Thus for every pair AgX and AhY there is a *unique* function $AfX \times Y$.

But $X \times Y$ is special in that it is "universal": for any *other* set A, if you give me functions $A \to X$ and $A \to Y$, I can use it build a *unique* function $A \to X \times Y$.

We can do this in any general category, defining a so-called product. Let X and Y be objects in any category $\mathbb{A}$. The product consists of an object $X \times Y$ and arrows π_X, π_Y to X and Y (thought of as projection). We require that for any object A and arrows AgX, AhY, there is a *unique* function $AfX \times Y$ such that the diagram

commutes. Strictly speaking, the product should consist of *both* the object $X \times Y$ and the projection maps π_X and π_Y. However, if π_X and π_Y are understood, then we often use $X \times Y$ to refer to the object, and refer to it also as the product.

Products do not always exist; for example, take a category with just two objects and no non-identity morphisms. Nonetheless: [Uniqueness of products] When they exist, products are unique up to isomorphism: given two products P_1 and P_2 of X and Y there is an isomorphism between the two objects.

Proof. This is very similar to the proof that initial objects are unique up to unique isomorphism. Consider two such objects P_1 and P_2, and the associated projection maps. There are unique morphisms f and g between P_1 and P_2 that make the entire diagram commute, according to the universal property.

On the other hand, look at $g \circ f$ and focus on just the outer square. Observe that $g \circ f$ is a map which makes the outer square commute, so by the universal property of P_1 it is the only one. But $_{P_1}$ works as well. Thus $_{P_1} = g \circ f$. Similarly, $f \circ g =_{P_2}$ so f and g are isomorphisms. ■

We have only showed the objects P_1 and P_2 are isomorphic and did not make any assumption about the projection maps But I haven't (and won't) define isomorphism of the entire product, and so in what follows if I say "P_1 and P_2 are isomorphic" I really just mean the objects are isomorphic.

the f and g above are the only isomorphisms between the objects P_1 and P_2.

The nice fact about this "universal property" mindset is that we don't have to do anything to show existence the "universal property" allows us to bypass all this work by saying "the object with these properties is unique up to unique isomorphism", thus we don't need to understand all the internal machinery to use these properties.

Of course, that's not to say we can't give concrete examples. [Examples of products]

[(a)] In Set, the product of two sets X and Y is their Cartesian product $X \times Y$. In Grp, the product of G, H is the group product $G \times H$. In $Vect_k$, the product of V and W is $V \oplus W$. In $CRing$, the product of R and S is appropriately the ring product $R \times S$. Let $\mathcal{P}$ be a poset interpreted as a category. Then the product of two objects x and y is the greatest lower bound; for example,If the poset is $(\mathbb{R}, \leq)$ then it's $\min\{x, y\}$. If the poset is the subsets of a finite set by inclusion, then it's $x \cap y$. If the poset is the positive integers ordered by division, then it's $\gcd(x, y)$.

We can also further this by extending these definitions to more than one object Consider a set of objects $(X_i)_{i \in I}$ in a category $\mathbb{A}$. We define a cone on the X_i to be an object A with some "projection" maps to each X_i. Then the product is a cone P which is "universal" in the same sense as before: given any other cone A there is a unique map $A \to P$ making the diagram commute. In short, a product is a "universal cone".

See also prob:associative$_product.$

One can also do the dual construction to get a coproduct: given X and Y, it's the object $X + Y$ together with maps $X \iota_X X + Y$ and $Y \iota_Y X + Y$

such that for any object A and maps XgA, YhA there is a unique f for which

commutes. We'll leave some of the concrete examples as an exercise this time, for example:

4 Monic and epic maps

The notion of "injective" doesn't make sense in an arbitrary category since arrows need not be functions. The correct categorical notion is: A map XfY is monic (or a monomorphism) if for any commutative diagram A ^{g} $_h X^f Y$ we must have $g = h$. In other words, $f \circ g = f \circ h \implies g = h$.

In most but not all situations, the converse is also true.

More generally, as we said before there are many categories with a "free" object that you can use to think of as elements. An element of a set is a function $1 \to S$, and element of a ring is a function $\mathbb{Z}[x] \to R$, et cetera. In all these categories, the definition of monic literally reads 'f is injective on $\operatorname{Hom}_{\mathbb{A}}(A, X)$". So in these categories, 'monic' and "injective" will coincidde

That said, here is the standard counterexample. An additive abelian group $G = (G, +)$ is called *divisible* if for every $x \in G$ and $n \in \mathbb{Z}$ there exists $y \in G$ with $ny = x$. Let $DivAbGrp$ be the category of such groups. Show that the projection $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is monic but not injective.

Of course, we can also take the dual notion. A map XfY is epic (or an epimorphism) if for any commutative diagram

we must have $g = h$. In other words, $g \circ f = h \circ f \implies g = h$.

Note here thatSurjective $\implies$ epic.

However, there are some cases when it does fail. Here are some claassic examples [Epic but not surjective]

[(a)] In $CRing$, the inclusion $\mathbb{Z}\mathbb{Q}$ is epic but not surjective Indeed, if two homomorphisms $\mathbb{Q} \to A$ agree on every integer then essentiallly they have to agree everywhere In the category of *Hausdorff* topological spaces in fact epic $\iff$ dense image (like $\mathbb{Q}\mathbb{R}$).

Thus failures arise when a function $f : X \to Y$ can be determined by just some of the points of X.

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