# Draft - Further Classical Topics In Finite Group Theory

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## 1 Introduction

Nilpotent groups are an interesting class of groups that are ”intuitively almost abelian”. There are many interesting concepts that arise out of the study of nilpotent groups, such as the Frattini subgroup.

Nilpotent groups are most often studied in the context of Galois theory and in the study of Lie groups. They are based off of the idea of a ”central series” which essentially generates an abelian group from a nonabelian one. In this paper, we discuss the definition of nilpotent groups and the interesting results which arise out of their study.

## 2 Central Series

First, it is important to define *central series*, which make up the definition of nilpotent groups. Central series, as the name implies, deal with the center of each successive subgroup. Groups need not have central series - if a group has a central series, then it is nilpotent.

**Definition 2.1.** A central series of a group $G$ is a subgroup sequence $e = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$, such that $G_i \lhd G$ and $G_{i+1}/G_i \leq Z(G/G_i)$ or $[G, G_{i+1} \leq G_i$ for $i$ such that $0 \leq i \leq n$. [RD]

The two equivalent parts of this definition can also be described in terms of lower and upper central series - both separate ways of describing the same concept.

**Definition 2.2.** A lower central series is the sequence of subgroups of a group $G$ ending in $e$ where $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [\gamma_i(G), G]$ for $i \geq 1$, or the commutator subgroup of $\gamma_i(G)$ and $G$. [RD]

We define $[g, h] = ghg^{-1}h^{-1}$. $[\gamma_i(G), G]$ is the commutator subgroup, or the group generated by all products of elements $\langle\, [a, b] \,|\, a \in \gamma_i(G), b \in G\,\rangle$. This definition means that the commutator subgroup is the extent to which a group fails to be abelian - the larger it is, the less abelian the group, since less elements of the group commute with each other. For abelian groups, the commutator subgroup is just $\{e\}$, since all elements commute with each other and therefore $ghg^{-1}h^{-1}$ is $e$ for all elements $g, h \in G$.

**Definition 2.3.** An upper central series of a group $G$ is a subgroup sequence $e = Z_0 \lhd Z_1 \lhd \ldots \lhd Z_n = G$, such that $Z_1 = Z(G)$ and $Z_{i+1}/Z_i$ is central in the quotient group $G/Z_i$ for $0 \leq i \leq n$.

The center of a group $G$, or $Z(G)$, is the set of elements which commute with all other elements of $G$. For upper central series, each successive subgroup $G_i$ must be normal in $G$, and $G_{i+1}/G$ must be the center of $G/G_i$. This definition ties in with the "almost abelian" aspect of nilpotent groups, since the center of a group is the set of all elements which commute with all other elements of a group.

All three definitions are equivalent and acceptable in defining nilpotent groups.

# 3 Nilpotent groups

**Definition 3.1.** A group $G$ is a Nilpotent group if and only if it has a central series of finite length. [Dic]

Since central series, lower central series, and upper central series all describe the same concept, we can give similar definitions for lower and upper central series. A group is also nilpotent if it lower central series that eventually reaches the trivial subgroup $e$. Similarly, a group is nilpotent if it has an upper central series terminating in $G$.

**Definition 3.2.** A group $G$ has nilpotency class $c$ where $\gamma_{c+1} = e$. [Dic]

The nilpotency class of a nilpotent group is the minimal length of its lower and upper central series. A nilpotent group with nilpotency class of at most $n$ is also called a nil-n group.

**Theorem 3.3.** *Every abelian group is nilpotent.*

*Proof.* The lower central series of any abelian group starts with $\gamma_1(G) = G$. As mentioned before, the commutator of an abelian group is $e$ and so $\gamma_2(G) = \{e\}$ . So, $G$ has a lower central series of finite length. Therefore, any abelian group is nilpotent with nilpotency class 1. [Bad] □

As an example of a nonabelian nilpotent group, we take $Q_8$, or the quaternion group.

The quaternion group is a group with 8 elements, consisting of the elements $1, -1$ as well as $i, j, k, -i, -j, -k$, which are all square roots of $-1$. Furthermore, $ij = k, ji = -k, jk = i, kj = -i, ik = j, ki = -j$. The rest of the relations are obtained from these.

We show this group to be nilpotent through constructing a lower central series. $\gamma_1(G) = G = Q_8$. The commutator subgroup of $G$ is $\gamma_2(G) = \{-1, 1\}$. The commutator subgroup of this group is $\gamma_3(G) = \{1\}$, the identity, meaning the group is nilpotent with nilpotency class 2.

We can also construct an upper central series for $Q_8$. As in the definition, $Z_1 = Z(Q_8) = \{1, -1\}$. To find $Z_2$, we find the group such that $Z_2/Z_1$ is central in $G/Z_1 = Q_8/\{1, -1\} = \{\{1, -1\}, \{i, -i\}, \{j, -j\}, \{k, -k\}\}$. The center of this group is the group itself $(G/Z_1)$ - every element commutes with every other element. Therefore, $Z_2 = G$, meaning the group is nilpotent, since it terminates in $G$. This also shows that it has a nilpotency class of 2.

**Theorem 3.4.** *Finite p-groups are nilpotent.*

*Proof.* A $p$-group is a group in which all elements (and correspondingly the group itself) have orders which are a power of $p$.

Consequently, $G/Z_i$ is a p-group as well (since the index of $Z_i$ must be a power of $p$ and therefore the order of the quotient group is also a power of $p \to$ it is a $p$-group.

Another property of p-groups is that they have a nontrivial center. That is, for a p-group $G$, $Z(G) \neq \{e\}$. Therefore, by this theorem, $Z(G/Z_i)$ is nontrivial. Therefore, $Z(G/Z_i) \subseteq Z(G/Z_{i+1})$ for $i$ and $Z(G/Z_i) \neq Z(G/Z_{i+1})$. □

Example: We show that the dihedral group $D_4$ with order 8 is nilpotent using lower central series. The commutator subgroup of $D_4$ is $\gamma_1(G) = \{e, \rho^2\}$. To obtain $\gamma_2(G)$ we compute $[\gamma_1(G), G]$. This is equal to $\{e\}$: for $g, h$ in these respective groups, $[g, h] = ghg^{-1}h^{-1} = e$, because both $e$ and $\rho^2$ commute with all elements of the group. Thus, $\gamma_2(G) = \{e\}$ and the group $D_4$ has a lower central series, making it nilpotent.

**Theorem 3.5.** *The direct product of two nilpotent groups is nilpotent.*

*Proof.* To begin the proof, we define $G = H \times K$, where $H$ and $K$ are nilpotent. To construct a lower central series, we can use the fact that $[H \times K, H \times K] = [H, H] \times [K, K] = H_1 \times K_1$ and $[H_1 \times K_1, H \times K] = [H_1, H] \times [K_1, K]$ and so on for all $i$ for which there is $H_i$ and $K_i$. We denote $G_i = [G_{i-1}, G]$. If we show that $G_i \leq H_i \times K_i$ for all $i$, we will be able to show that $G_i$ eventually reaches the trivial group $e$.

We can do this inductively: for the base case, we already know that $G_1 = [H \times K, H \times K] = H_1 \times G_1$. Next, we know that, by the inductive hypothesis, $G_i \leq [H_{i-1} \times K_{i-1}, H \times K] \leq [H_{i-1}, H] \times [K_{i-1}, K] = H_i \times K_i$. Having inductively proved that $G_i \leq H_i \times K_i$, and knowing that $H$ and $K$ are nilpotent, since $H_i$ and $K_i$ eventually go to $e$, $G_i$ must also eventually become $e$, meaning $G$ is nilpotent.[Gar] □

**Theorem 3.6.** *A finite group is nilpotent if every Sylow subgroup of it is normal.*

# 4 Maximal Subgroups

**Definition 4.1.** A subgroup $H$ of a group $G$ is maximal if it is a proper subgroup such that no proper subgroup $K$ exists where $H$ is a proper subset of $K$. [Bad]

This definition doesn't just include subgroups of the highest order - it allows for multiple maximal subgroups, since some subgroups may exist which are not contained within the subgroup(s) of highest order.

**Definition 4.2.** A maximal normal subgroup $N$ of $G$ is a maximal proper subgroup which is also normal, and the only normal subgroup containing it is the entire group. [Bad]

Equivalently, $N$ must be normal and $G/N$ must be a simple group. [Gar]

**Theorem 4.3.** *A normal subgroup $N$ of a group $G$ is a maximal normal subgroup iff the quotient $G/N$ is a simple group.*

*Proof.* This essentially follows from the 4th isomorphism theorem (or the correspondence theorem), which states that for a group $G$ and a normal subgroup $N$, there is a bijection between the set of all subgroups of $G$ containing $N$ and the subgroups of the quotient group $G/N$. If the group $G/N$ is simple, then, since it has no subgroups, the subgroup is not contained in any other groups, making it a maximal normal subgroup. [Iso] □

Next, we look at an example of maximal normal subgroups, namely the maximal normal subgroups of $S_4$. The nontrivial normal subgroups of $S_4$ are $A_4$ and normal $V_4$, and of these, only $A_4$ is maximal.

# 5 The Frattini Subgroup

**Definition 5.1.** The Frattini Subgroup of a group $G$ (denoted $\Phi(G)$) is a (normal) subgroup which is the intersection of all maximal subgroups of $G$.

The Frattini subgroup is a concept first introduced by Giovanni Frattini in 1885, defined as the intersection of the maximal subgroups of a group $G$.

**Remark 5.2.** Frattini's argument for Sylow $p$-subgroups: States that for a normal subgroup $H$ of a group $G$ and for $P$, a Sylow $p$-subgroup of $H$, $G = N_G(P)H$, where $N_G(P)$ is the normalizer of $P$ in $G$.

**Definition 5.3.** The normalizer of a subgroup $P$ in group $G$ is the set of elements of $G$ which commute with the subgroup $P$. Equivalently, it is the largest subgroup $K$ ($P \leq K \leq G$) in which $P$ is normal.

*Proof.* We prove that any element of $g$ is in $N_G(P)H$. First, we conjugate both sides of the relation $P \leq H$ to get $g^{-1}Pg \leq g^{-1}Hg = H$, since $H$ is normal in $G$. Because $g^{-1}Pg \leq H$ and its order is the same as $P$ (since conjugation is injective/on-to-one), the group $g^{-1}Pg$ is also a Sylow $p$-subgroup of $H$.

Since Sylow $p$-subgroups are conjugate by the second Sylow theorem, we can conjugate $P$ once again by $h \in H$ to get $h^{-1}Ph = g^{-1}Pg$. Next, we conjugate both sides by $h \in H$ and see that $P = hg^{-1}Phgh^{-1}$. Now, we know that, due to the definition of a normalizer, $gh^{-1} \in N_G(P)$, since it commutes with $P$. Therefore, multiplying by $h$ on both sides yields $g \in N_G(P)h$ for all $g \in G$, meaning that $G = N_G(P)H$. □

The Frattini argument is an important part of Frattini's work - it can also be applied to automorph-conjugate subgroups, which are subgroups such that the subgroups they are mapped to via automorphism are also conjugate to the subgroup. We will use the Frattini argument to prove the next theorem - that the Frattini subgroup is nilpotent.

**Theorem 5.4.** *The Frattini Subgroup is the subset of all non-generators of $G$.*

**Definition 5.5.** A non-generating element of a group $G$ is an element of the group that is not required to be part of the generating set, which is the set of elements of $G$ from which all elements of the group can be obtained.

The proof of this theorem is slightly beyond the scope of this paper, as it requires Zorn's lemma and other parts of set theory, but it is needed to prove the next theorem, that the Frattini Subgroup is nilpotent.

**Theorem 5.6.** *The Frattini Subgroup of a finite group $G$ is nilpotent.*

*Proof.* We know that a finite group is nilpotent if every Sylow $p$-subgroup of it is normal. Therefore, we use the Frattini Argument to prove that every Sylow $p$-subgroup of the Frattini Subgroup is normal.

Since $\Phi(G) \lhd G$, by the Frattini Argument, $\Phi(G)N_G(P) = G$ for any Sylow $p$-subgroup of $G$. By Theorem 5.5, since the Frattini subgroup is all nongenerators of the group, we can "delete" its elements from the product and still end up with the whole group. Therefore $N_G(P) = G$. Since the normalizer of $P$ is the set of elements in the group that commute with $P$, because its $G$, $P$ commutes with the whole group, making it normal. This is true for any Sylow $p$-subgroup of $G$, meaning it is nilpotent. □

# References


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