Representation Theory

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This article, we'll discuss the representations of groups. The correct way to introduce the representation of a group is through a k-algebra. Many of the theorems and definitions for k-algebras are analogous to those of groups. We assume familiarity with standard linear algebra topics such as tensors, direct sums and trace. Unless otherwise stated, k refers to a field, V a vector space, A an algebra and G a group.

§1 Motivation

In group theory, we have Cayley's theorem, which tells us that given a finite group G, there is an injective map $G \to S_n$, where $n = |G|$. However, the symmetric group is *pretty weird* and difficult to work with. However, we would still like to be able to concretely work with groups and understand their properties through a "larger" group that contains all of them. It seems S_n is the only "larger" group we can use until we realize that we can actually construct a homomorphism from S_n to $GL_n(\mathbb{R})$. Here is the representation for S_3.

$$() \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (12) \to \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (23) \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$(31) \to \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad (123) \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad (132) \to \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

We can even generalize this to arbitrary n. Why should we care about this? *It tells us that any subgroup of S_n can also be expressed as elements of $GL_n(\mathbb{R})$. But by Cayley's theorem that means every group can be expressed as matrices.* This is the goal of representation theory. Make all the group elements into a matrix so that we can concretely understand them through linear algebra.

§2 Representations of k-Algebras and Groups

The natural setting for representation theory is a k-algebra. We want to express the elements of a k-algebra as matrices, and by doing so we'll be able to do the same to G for free. The most important thing to note is that matrices are really just linear operators on a vector space! *So a representation should actually be thought of as a map from a k-algebra to a set of functions.*

Definition 2.1. A *k-algebra* A is a k-vector space, equipped with associative multiplication $\cdot : A \times A \to A$ satisfying the following axioms

- $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$

- $a \cdot (\lambda b) = (\lambda a) \cdot b = \lambda \cdot (ab)$ for all $\lambda \in k$
- There exists an element 1_A, called the identity, such that $1_A a = a 1_A = a$ for all $a \in A$

Alternatively, A can be viewed as a possibly non-commutative ring with an embedding $k \hookrightarrow A$. Specifically, the image of k in A is $\{k1_A : k \in A\}$.

With either defintion, The set of polynomials $k[x]$ form a k-algebra with function composition, the identity being x. If V is a k-vector space then $\text{Mat}(V)$, the set of linear maps $V \to V$, also from a k-algebra under function composition.

Example 2.2 (Group Algebra)
For a group G, we define the group algebra

$$k[G] = \left\{ \sum_{i=1}^{n} c_i g_i : c_i \in k, g_1 \in G \right\}.$$

In other words, $k[G]$ is a k-vector space, with the basis elements provided by G. Multiplication works as expected:

$$(ag + bh)^2 = agag + agbh + bhag + bhbh = a^2 g^2 + ab(gh + hg) + b^2 h^2,$$

where $a, b \in k$ and $g, h \in G$.

The *group algebra* is one of the most useful k-algebras since it adapts the notion of a group to "fit" a k-algebra which means we can use k-algebra theorems on groups.

Definition 2.3. Given two k-algebras A and B, we define a *homomorphism* $F : A \to B$ such that

- F is a linear map when A and B are viewed as vector spaces.
- $F(ab) = F(a)F(b)$ for all $a, b \in A$, and $F(1_A) = 1_B$.

Now we are ready to define what a representation of an algebra is.

Definition 2.4. A representation of a k-algebra A is a pair (V, ρ), where V is a k-vector space and $\rho : A \to \text{Mat}(V)$ defines a k-algebra homomorphism.

From this, we can prove that ρ fixes k. Indeed, for any scalar λ, since ρ is linear, we have $\rho(\lambda) = \lambda\rho(1_A) = \lambda I$, where I is the identity in $\text{Mat}(V)$. The other way to think about a representation is an action of A on V.

Definition 2.5. Alternatively, a representation of a k-algebra is an action $\cdot : A \times V \to V$ of A on a k-vector space V satisfying the following axioms

- $(a + b) \cdot v = a \cdot v + b \cdot v$, $a \cdot (v + w) = a \cdot v + a \cdot w$, and $a \cdot (b \cdot v) = (ab) \cdot v$ for all $a, b \in A$ and $v, w \in V$.
- $\lambda \cdot v = \lambda v$ for all $\lambda \in k$.

In particular, the action is given by $a \cdot v = \rho(a)(v)$, where ρ is the homomorphism mentioned earlier. In discussing representations, the latter definition is used more often, and ρ is often omitted.

Example 2.6 (Representation into $\text{Mat}(V)$)

Let (V, ρ) be a representation of A. Then, $\text{Mat}(V)$ is also a representation of A, with the action given by $a \cdot T = \rho(a) \circ T$, where $\circ$ represents function composition. This satisfies all the conditions:

- $a \cdot (b \cdot T) = a \cdot (\rho(b) \circ T) = \rho(a) \circ \rho(b) \circ T = \rho(ab) \circ T = (ab) \cdot T$
- $a \cdot (T + S) = \rho(a) \circ (T + S) = \rho(a) \circ T + \rho(a) \circ S = a \cdot T + a \cdot S$
- $(a + b) \cdot T = \rho(a + b) \circ T = (\rho(a) + \rho(b)) \circ T = a \cdot T + b \cdot T$
- $\lambda \cdot T = \rho(\lambda) \circ T = \lambda I \circ T = \lambda T$,

where I denotes the multiplicative identity of $\text{Mat}(V)$.

Example 2.7 (Defining Representations of G)

In this example, we will motivate the definition for a representation of a group. Let V be a representation of $k[G]$ and consider the k-algebra homomorphism $F : k[G] \to \text{Mat}(V)$.

Consider what F does to the elements of G. Since the elements of G form a basis, where we send one element of G does not have an effect on where we send another. Thus, the fact that F is linear is irrelevant. Note that $F(gh) = F(g)F(h)$ for all $g, h \in G$. Therefore, the image of G under F is a *group*, since F restricted to G defines a *group* homomorphism. But that implies that $F(g)$ is invertible for all g! Hence, F really just defines a group homorphism $G \to GL(V)$. Conversely, given a homomorphism $G \to GL(V)$, we can extend it to a k-algebra homomorphism of $k[G]$ through linearity.

The above example yields the following natural definiton.

Definition 2.8. A representation of a group G is a pair (V, ρ) such that $\rho : G \to \text{GL}(V)$ defines a group homomorphism. Alternatively, a representation of G is an action of G on a vector space V. Conventionally, V is usually a vector space over $\mathbb{C}$.

Example 2.9 (Representation of D_4)

Consider a map $\phi : D_4 \to GL_2(\mathbb{R})$. As usual, let r and s denote rotation and reflection in D_4. We claim that

$$\phi(r) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } \phi(s) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

provides an embedding $D_4 \to GL_2(\mathbb{R})$. Indeed, by straightforward multiplication, we can verify that $\phi(r)^4 = \phi(s)^2 = 1$, and $\phi(r)\phi(s) = \phi(s)\phi(r)^{-1}$. Hence, the group presentation is preserved, and so the image of D_4 under ϕ is isomorphic to D_4. Thus, ϕ constitutes a representation of D_4.

Finally, given two representations V and W, we are able to combine them to make a ”larger” representation of A.

Definition 2.10. If (V, ρ_V) and (W, ρ_W) are two representations of a k-algebra A, then $V \oplus W$ is also a representation with the action given by $a \cdot (v, w) = (a \cdot v, a \cdot w)$. If V and W are subspaces of a larger space U, then the direct sum is addition, and the action is instead defined as $a \cdot (v + w) = a \cdot v + a \cdot w$. Naturally, we can extend this definition for groups as well.

In fact, with a little bit of linear algebra, we can say that

$$\rho(a) = \begin{bmatrix} \rho_V(a) & 0 \\ 0 & \rho_W(a) \end{bmatrix}$$

Indeed, this just follows from putting the coefficients of V (after expressing it in terms of the basis) on top and the coefficients of W on the bottom.

Example 2.11 (Representation of D_4 in $\mathbb{R}^4$)

We can take our earlier representation for D_4 and represent it in $\mathbb{R}^4 \cong \mathbb{R}^2 \oplus \mathbb{R}^2$, as well. From the above analysis, we can consider the map $\phi : D_4 \to GL_4(\mathbb{R})$ given by,

$$\phi(r) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } \phi(s) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

This is really just a "copy" of our earlier representation, so in a sense this is "reducible". That is, given this, we can figure out what the representations over $\mathbb{R}^2$ would look like. More on this in the next section.

Given two representations of an algebra, we have a direct sum operation, which makes it "larger". In the next section, we will aim to do the exact opposite of this. Given a representation, we will want to break it down into smaller ones, much like breaking a number down into its prime factors.

§3 Irreps and Schur's Lemma

For a group G, we have the notion of a subgroup, which formalizes the fact that sometimes a smaller subset of G can also form a group. For abelian groups, we were even able to characterize the parent group G from its subgroups. We hope to do something similar with representations.

Definition 3.1. Let W be a subspace of a representation V of A. We say that W is a *subrepresentation* if $a \cdot w \in W$ for every $a \in A$ and $w \in W$. In other words, the action shuffles the elements of W.

Definition 3.2. A representation V is said to be an *irrep (irreducible representation)* if it has no nontrivial subrepresentations. V is said to be *indecomposable* if there are no nontrivial subrepresentations U and W such that $V = U \oplus W$.

Definition 3.3. A representation V is said to be *completely reducible* if it can be broken down into a direct sum of irreducible subrepresentations. An algebra A is said to be *semisimple* if all of its representations are completely reducible.

Clearly, if a representation is irreducible it is automatically indecomposable. Unfortunately the converse of this turns out to be false: *there are representations that are indecomposable but not irreducible.* However, for the case of semisimple algebras and $k[G]$ this turns out to be true, as we will see in the next section.

Example 3.4 (Irreducible $\neq$ Indecomposable)

Let $V = \mathbb{R}^{\oplus 2}$ be a representation of $A = \mathbb{R}[x]$. The homomorphism ρ is determined by where it sends x and 1. Suppose

$$\rho(x) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

All subrepresentations of V are 1-dimensional. Let W be one such subrepresentation. So suppose $(a, b) \in W$. Then $W = \{(ca, cb) : c \in \mathbb{R}\}$. Now consider how x acts on (a, b). We have $x \cdot (a, b) = \rho(x)(a, b) = (a + b, b)$. But, we know that if W is a subrepresentation, there exists a constant c such that $(ca, cb) = (a + b, b)$. So either, $c = 1$ or $b = 0$. But both cases lead to $b = 0$, so the only subrepresentation of V is $W = \{(t, 0) : t \in \mathbb{R}\}$. Hence, V is not an irrep, but *is* indecomposable.

Going back to our second representation of D_4, we see that it has subrepresentations given by $\{(a, b, 0, 0) : a, b \in \mathbb{R}\}$ and $\{(0, 0, c, d) : c, d \in \mathbb{R}\}$. Therefore, it must be decomposable and hence reducible.

To figure out if two groups were "the same" we invented the homomorphism, and declared that they would be equivalent if the homomorphism was also a bijection. We want to be able to do something similar with representations.

Definition 3.5. Let V and W be two representations of A. A linear map $T : V \to W$ is said to be a *morphism* if

$$T(a \cdot v) = a \cdot T(v)$$

for all $a \in A$ and $v \in V$. The morphism is said to be an isomorphism if T is bijective. We also define the subspaces $\ker T = \{v \in V : T(v) = 0\}$ and $\operatorname{im} T = \{T(v) : v \in V\}$. Finally, we let $\operatorname{Hom}_{\text{rep}}(V, W)$ denote the set of all morphisms from V to W. It forms a k-algebra with function composition.

Example 3.6 ($\operatorname{Mat}(V) \cong V^{\oplus \dim V}$)

Let d be the dimension of V, and let $\beta_1, ..., \beta_d$ be a basis for V. $\operatorname{Mat}(V)$ contains linear operators which are determined by where they send the basis elements. So, pick $T \in \operatorname{Mat}(V)$ and consider the map $S : \operatorname{Mat}(V) \to V^{\oplus d}$ given by $S(T) = (T(\beta_1), ..., T(\beta_r))$.

The map is obviously bijective since there's no restriction on where you can send the basis elements. To show its a morphism, we see

$$S(a \cdot T) = S(\rho(a) \circ T) = (\rho(a)(T(\beta_1)), ..., \rho(a)(T(\beta_r))) = \rho(a) \circ S(T) = a \cdot S(T),$$

so we are done.

From your intuition in group theory, you may have already guessed the following theorem.

Theorem 3.7

Let A be a k-algebra and let V and W be representations of A. If T is a morphism from V to W, $\ker T$ is a subrepresentation of V and $\operatorname{im} T$ is a subrepresentation of W.

Proof. Select $v \in \ker T$. Then, $T(a \cdot v) = a \cdot T(v) = 0$, so $\ker T$ remains invariant under the action of A. The proof for $\operatorname{im} T$ is essentially the same. □

From this, we more or less get Schur's Lemma for free.

Lemma 3.8 (Schur's Lemma)
Let V and B be representations of a k-algebra A. If $T : V \to W$ is a nonzero morphism.

- If V is an irrep, then T is injective.
- If W is an irrep, then T is surjective.
- If both are irreps, then T is an isomorphism.

In the case that k is algebraically closed, we can actually charactarize every single morphism $T : V \to V$.

Lemma 3.9 (Schur's Lemma + Algebraically closed)
Suppose k is algebraically closed. Let V be a representation of k-algebra A, and $T : V \to V$ be a morphism. Then there is a $\lambda \in k$ such that $T(v) = \lambda v$.

Proof. First, since k is algebraically closed, it had an eigenvalue λ. Consider the map $F = T - \lambda I$, where I is the identity matrix. Suppose F is nonzero. The kernel of this is clearly non-trivial since there is an eigenvalue. But by Schur's lemma, since V is an irrep, the kernel must be trivial, contradiction. □

Example 3.10 (Characterizing $\operatorname{Hom}_{\text{rep}}(V, V^{\oplus m})$ for irreps V)
First, note that Lemma 3.9 gives $\operatorname{Hom}_{\text{rep}}(V, V) \cong k$. We will prove that

$$\operatorname{Hom}_{\text{rep}}(V, V^{\oplus m}) \cong k^{\oplus m}$$

provided k is algebraically closed. But to specify a morphism $T : V \to V^{\oplus m}$, we just need to specify m different morphisms $V \to V$ (one for each component of $V^{\oplus m}$). But Lemma 3.9 tells us that each of these morphisms is just multiplication by scalar. Therefore, the only morphisms $V \to V^{\oplus m}$ are of the form $T(v) = (c_1 v, \ldots, c_m v)$. Naturally, this is isomorphic to $k^{\oplus m}$.

§4 Density and Maschke

Now, we are ready to introduce the Density Theorem, which we state without proof.

Theorem 4.1 (Jacobson Density Theorem)

If $V_1, V_2, ..., V_r$ are non-isomorphic finite dimensional representations of A, then there is a surjective map

$$\rho : A \to \bigoplus_{i=1}^{r} \mathrm{Mat}(V_i),$$

provided k is algebraically closed.

Notice the relation to the Chinese Remainder Theorem. If $(M_1, ..., M_r)$ was a tuple of pairwise relatively prime integers, and we picked a tuple $(q_1, ..., q_r) \in \mathbb{Z}/M_1\mathbb{Z} \times \cdots \times \mathbb{Z}/M_r\mathbb{Z}$, then we can find an integer $N \in \mathbb{Z}/(M_1 \cdots M_r\mathbb{Z})$ such that $N = q_i$ in $\mathbb{Z}/M_i\mathbb{Z}$.

Similarly, if I give you a tuple of the form $(T_1, ..., T_r)$, where $T_i \in \mathrm{Mat}(V_i)$ you can find an a such that $(\rho_1(a), ..., \rho_r(a)) = (T_1, ..., T_r)$.

The Density Theorem helps us put bounds on the number of irreps a k-algebra can have. More specifically,

Lemma 4.2

The number of irreps of A cannot exceed $\dim A$.

Proof. Suppose there are r such representations, labeled V_1, ..., V_r. First, we note that $\mathrm{Mat}(V_i) \cong V_i^{\oplus \dim V_i}$ as representations of A. Therefore, by the Density Theorem

$$\dim A \geq \sum_{i=1}^{r} (\dim V_i)^2 \geq r$$

□

Of course, we are also interested in when equality occurs. For this, we need the charactarization theorem for semisimple algebras, which we state without proof.

Theorem 4.3 (Semisimple Algebras)

A is semisimple if and only if $A \cong \bigoplus_i \mathrm{Mat}_{d_i}(k)$, where $\mathrm{Mat}_{d_i}(k)$ consists of the square $d_i \times d_i$ matrices with entries in k.

Combined with Lemma 3.2, we obtain the following theorem.

Theorem 4.4

A is semisimple if and only if $\sum_i (\dim V_i)^2 = \dim A$.

Proof. By the Density Theorem, equality can only hold when the map $\rho : A \to \bigoplus_{i=1}^{r} \mathrm{Mat}(V_i)$ is bijective. Hence, equality holds if and only if $A \cong \bigoplus_{i=1}^{r} \mathrm{Mat}(V_i) \cong \bigoplus_{i=1}^{r} \mathrm{Mat}_{d_i}(k)$, where d_i is the dimension of V_i. But this implies A is semisimple. □

Now we are ready to state a fundamental connection between representations of groups and aglebras.

Theorem 4.5 (Maschke's Theorem)

Let G be a finite group, and k a field whose characteristic does not divide $|G|$. The algebra $k[G]$ is semisimple.

Proof. To show that it is semisimple, we need to show that given any reducible finite dimensional representation V can decompose it into two subrepresentations. Let W be a non-trivial subrepresentation of V. We claim that there is a U such that $V = U \oplus W$.

Let $\{w_1, ..., w_p\}$ be a basis for W. We can extend this to a basis for V, so let $\{w_1, ..., w_p, u_1, ..., u_q\}$ be a basis for V. Finally, let U be the vector space spanned by the u_i's. It follows that $V = U \oplus W$, so all we need to do now is show U is also a subrepresentation.

Consider the projection $\pi : V \to W$ given by $\pi(u + w) = w$, where we first express every v uniquely as $u + w$ for $u \in U$ and $w \in W$. The trick is to consider the following map

$$P(v) = \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot v).$$

The map has the following properties:

- $P(w) = w$ for all $w \in W$
- $P(v) \in W$ for all $v \in V$
- P is a morphism.

To prove the first property, note that since W is a subrepresentation, we have $g^{-1} \cdot w \in W$, so $P(w) = \frac{1}{|G|} \sum_{g \in G} g(g^{-1} \cdot w) = \frac{1}{|G|} \sum_{g \in G} w = w$.

The second property follows from the fact that $\pi(g^{-1} \cdot v) \in W$, so in particular $g \cdot \pi(g^{-1} \cdot v)$ is always an element of W. Therefore, $P(v)$ is merely the sum of elements in W, multiplied by a scalar, so it must lie in W.

To show P is a morphism on $k[G]$, we only need to show that it is a morphism on the basis, i.e. G. So pick an element $h \in G$, and note

$$h^{-1} \cdot P(h \cdot v) = \frac{1}{|G|} \sum_{g \in G} h^{-1} g \cdot \pi(g^{-1} h \cdot v) = \frac{1}{|G|} \sum_{k \in G} k \cdot \pi(k^{-1} \cdot v) = P(v).$$

where the last step follows from noting as g goes through G, $k = h^{-1} g$ also goes through G. Therefore, multiplying both sides by h, we see P is a morphism.

From the properties mentioned above, we see that $P(P(v)) = v$ for all $v \in V$, i.e. P is idempotent. Hence, $V = \ker P \oplus \operatorname{im} T = \ker P \oplus W$. Thus, V is decomposable, and we are done.

□

§5 Everyone Loves Characters

Every possible representation can be expressed as a matrix. Thus, it makes sense to try and understand linear algebra properties of these maps. A natural candidate for this is trace. Surprisingly this turns out to be a great choice since as we will see shortly, it ties in naturally with groups.

§5.1 Characters I: k-Algebra Perspective

There isn't a whole lot to say here, and all the properties listed below carry over to groups as well.

Definition 5.1. Let A be a k-algebra, and V a representation. Then we define the *character* $\chi_V : A \to k$ by $\chi_V(a) = \mathrm{Tr}(\rho(a))$.

We have the following basic results about trace.

Theorem 5.2

Let A be an algebra, and suppose V and W are two different representations.

- $\chi_V(1_A) = \dim V$
- $\chi_{V\oplus W} = \chi_V + \chi_W$
- $\chi_V(ab) = \chi_V(ba)$ for all $a, b \in A$
- χ_V is a linear map.

Proof. We have $\chi_V(1_A) = \mathrm{Tr}(\rho(1_A)) = \mathrm{Tr}(I)$, where I is the identity on $\mathrm{Mat}(V)$. So it follows that $\chi_V(1_A) = \dim A$.

We remarked earlier that the homomorphism $A \to \mathrm{Mat}(V \oplus W)$ is given by the matrix $\rho(a) = \begin{bmatrix} \rho_V(a) & 0 \\ 0 & \rho_W(a) \end{bmatrix}$. The second part is direct from this.

The third part follows from noting that $\mathrm{Tr}(ST) = \mathrm{Tr}(TS)$ for any matrices T and S. The last part is immediate since trace itself is linear. □

Continuing in the spirit of our earlier examples, let's compute the trace of each element in D_4!

Example 5.3 (Characters of D_4)

We'll compute the character of our first representation of D_4 in 2×2 matrices.

g	1	r, r^3	r^2	s, sr^2	sr, sr^3
$\chi(g)$	2	0	-2	0	0

We've conveniently broken down the elements of D_4 into their conjugacy classes. As you may have noticed the character seems to be the same for each element of a given conjugacy class. This is no coincidence, as we will see in the next section.

§5.2 Characters II: Group perspective

From here on out, we will set $k = \mathbb{C}$, and we will work on specializing all of the above results for groups. Characters for groups are defined in exactly the same way, so I won't bother redefining it.

Definition 5.4. $\mathrm{Class}(G)$ is the set of conjugacy classes in G.

Why would we want to consider conjugacy classes of G? Well, Theorem 5.2 shows that $\chi(ghg^{-1}) = \chi(gg^{-1}h) = \chi(h)$. *So χ treats all elements of a conjugacy class with equal disrespect!* In particular, we can think of χ as a function from $\mathrm{Class}(G) \to \mathbb{C}$, since χ

only depends on *which* conjugacy class you choose. Furthermore, it turns out that there are exactly $|\operatorname{Class}(G)|$ irreps of G (see [1] for proof).

Definition 5.5. Let G be a group with representations V and W. Then the tensor product $V \otimes W$ can be made into a representation of G by imposing the action $g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w)$.

You might wonder why we can't define the same for a k-algebra. The reason is that A *has additional distributive structure that G doesn't.* More specifically, we can distribute over elements of A like $(a + b) \cdot (v \otimes w) = a \cdot (v \otimes w) + b \cdot (v \otimes w)$, which isn't compatible with the group-theoretic definition of tensor representation. To see this, we can simply take $a = b = 1_A$.

Definition 5.6. Let G be a group with representation V, and consider its dual space $V^\vee$. We can make $V^\vee$ into a representation by taking $f \in V^\vee$ and sending it to $g \cdot f$, where $g \cdot f$ in $V^\vee$ is the function providing the map $v \to f(g^{-1} \cdot v)$.

This definition also fails in a k-algebra, since we do not require all elements to have inverses.

Definition 5.7. We define $\mathbb{C}_{\text{Class}}(G)$ to be the set of functions $\operatorname{Class}(G) \to \mathbb{C}$ treated as a $\mathbb{C}$-vector space. We make this into an inner product space by definining

$$\langle f, h \rangle = \frac{1}{|G|} \sum_{g \in G} f(g) \overline{h(g)}$$

The reason for defining this is the following theorem:

Theorem 5.8

If V and W are representations of G, then

$$\langle \chi_V, \chi_W \rangle = \dim \operatorname{Hom}_{\text{rep}}(W, V).$$

Moreover, the characters of the irreps of G form an orthonormal basis.

The proof can be found in [1]. Additionally, here are some properties of characters that apply solely to groups.

Theorem 5.9

If V and W are representations of G, we have

- $\chi_{V \otimes W}(g) = \chi_V(g)\chi_W(g)$
- $\chi_{V^\vee}(g) = \overline{\chi_V}(g)$

Proof. The first property falls out from the well known trace identity $\operatorname{Tr}(T \otimes S) = \operatorname{Tr}(T)\operatorname{Tr}(S)$. For concreteness, we spell it out

$$\chi_V(g)\chi_W(g) = \operatorname{Tr}(\rho_V(g))\operatorname{Tr}(\rho_W(g)) = \operatorname{Tr}(\rho_V(g) \otimes \rho_W(g)) = \operatorname{Tr}(\rho_{V \otimes W}(g)) = \chi_{V \otimes W}(g).$$

The proof of the second theorem is omitted, but can be found in [1]. □

From Theorem 5.8, we can actually derive a criterion for a representation to be an irrep.

Theorem 5.10

If V is an irrep of a group G over $\mathbb{C}$, then $\langle \chi_V, \chi_V \rangle = 1$.

Proof. In the section 3, we derived that $\mathrm{Hom}_{\mathrm{rep}}(V, V) \cong \mathbb{C}$. Hence, $\dim \mathrm{Hom}_{\mathrm{rep}}(V, V) = 1$, and it follows that $\langle \chi_V, \chi_V \rangle = 1$. □

§5.3 Character Tables

For aesthetic purposes, we can represent all of our information in a cute table.

Example 5.11 (Character Table for D_3)

Let us construct the character table for D_3. There are 3 conjugacy classes, and hence 3 irreps. Suppose these irreps have dimension d_1, d_2 and d_3. Then, since $\mathbb{C}[G]$ is semisimple, we have

$$d_1^2 + d_2^2 + d_3^2 = 6 \implies \{d_1, d_2, d_2\} = \{1, 1, 2\}.$$

Here are the resulting representations:

- $\mathbb{C}_{\mathrm{triv}}$: The trivial representations that sends each element of D_3 to $[1]$ (the one dimensional identity matrix).
- $\mathbb{C}_{\mathrm{sgn}}$: The sign representations that sends $r \to [1]$ and $s \to [-1]$.
- V: The two dimensional representations given by $r \to \begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix}$ and $s \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, where $\omega = e^{2\pi i/3}$.

Here is the character table:

D_3	1	r, r^2	sr^j
$\mathbb{C}_{\mathrm{triv}}$	1	1	1
$\mathbb{C}_{\mathrm{sgn}}$	1	1	-1
V	2	-1	0

At the top, we have the conjugacy classes, and the numbers inside the chart are the values of the characters on those classes.

We can verify Theorem 5.8 for this special case. Indeed,

$$\langle \mathbb{C}_{\mathrm{sgn}}, V \rangle = \frac{1}{6}(1 \cdot (1)(2) + 2 \cdot (1)(-1) + 3 \cdot (-1)(0)) = 0$$

$$\langle V, V \rangle = \frac{1}{6}(1 \cdot (2)(2) + 2 \cdot (-1)(-1) + 3(0)(0)) = 1,$$

as expected.

Example 5.12 (Character Table for D_4)

As we saw in the earlier example, D_4 has 5 conjugacy classes. Following the same logic as before, we deduce that there are four irreps of dimensions 1 and one irrep of dimension 2. The irrep of dimension two is shown in Example 2.9.

It remains to find all the 1-dimensional irreps. Since they are one dimensional, we can simply think of them as homomorphism $D_4 \to \mathbb{C}^\times$. Suppose $r \mapsto a$ and $s \mapsto b$ for some $a, b \in \mathbb{C}^\times$. Then, we know that since $s^2 = 1$, we have $b = \pm 1$. Next, since $rs = sr^{-1}$, we know that $a^2 = 1$ (recall $\mathbb{C}^\times$ is abelian).

However, we know that there are 4 irreps. Hence, each choice of a and b provides a valid irrep. With this information, we can easily construct the character table.

D_4	1	r, r^3	r^2	s, sr^2	sr, sr^3
$V_1(1,1)$	1	1	1	1	1
$V_1(1,-1)$	1	1	1	-1	-1
$V_3(-1,1)$	1	-1	1	1	-1
$V_4(-1,-1)$	1	1	-1	-1	1
W	2	0	-2	0	0

The ordered pairs next to the V_i denote the pair (a, b) used to construct the representation.

References

[1] Evan Chen, *Napkin*. https://venhance.github.io/napkin/Napkin.pdf.

[2] Julie Linman, *Burnside's Theorem*. http://sites.science.oregonstate.edu/~swisherh/JulieLinman.pdf.