POLYA'S ENUMERATION THEOREM

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Abstract. In this paper, we examine some basic group theory and Burnside's Lemma. Burnside's Lemma leads to the broader Polya's Enumeration Theorem, which ha s interesting applications in coloring problems.

1. Group Theory Basics

Definition 1.1. [Jin] A **group** is a set G with an operation * that satisfies associativity, identity, and inverses:

- Associativity: For any $a, b, c \in G$, $ab(c) = a(bc)$.
- Identity: There exists identity element e such that $eg = ge$.
- Inverses: For any $g \in G$, g^{-1} exists such that $gg^{-1} = e$.

Definition 1.2. Let G be a group and X be a set. A **group action** is a function $\phi : G \times X \to X$ satisfying $\phi(e, x) = x$ and $\phi(g, \phi(h, x)) = \phi(gh, x)$.

Definition 1.3. Let G be a group acting on X. The **orbit** of an element $x \in X$ is the set $G.x = \{gx | g \in G\}$. The orbits partition the set X. The set of orbits over X is denoted X/G. The **stabilizer** of $x \in X$ is $G_x = \{g \in G : gx = x\}$.

Definition 1.4. For an element $g \in G$, the **fixed point** of X is an element $x \in X$ such that $gx = x$, denoted by X^g.

Definition 1.5. Let G be a group and $H \leq G$ a subgroup, and let $g \in G$ be some element. Then the set $gH = \{gh : h \in H\}$ is the **left coset** of H, and Hg is a **right coset** of H.

2. Burnside's Lemma

Given G is a group acting on set X, Burnside's Lemma states

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

Proof. [Zha] First we prove a related theorem:

Theorem 2.1. ***Orbit-Stabilizer Theorem*** *Let G be a finite group of permutations on set X. The orbit-stabilizer theorem gives $|G| = |G_x| * |G.x|$.*

Proof. For element $x \in X$, let $G.x$ be the orbit of x, and G_x be the stabilizer of x. Let L_x be the set of left cosets of G_x. The function f is defined $f : G.x \to L_x$.

We have that f is surjective by definition, because it is defined to be the function from $G.x$ to the left coset.

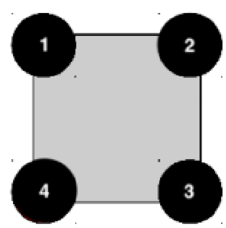
We also have $gG_x = g'G_x$, then for some $h \in G_x$, $g = g'h$. This means $gx = (g'h)x = g'(hx) = g'x$, which gives f is injective. Hence, we have the function f is a bijection, which proves Orbit-Stabilizer. ■

Thus, we have

$$\begin{aligned} |X/G| &= \sum_{x \in X} \frac{1}{|G.x|} \\ &= \sum_{x \in X} \frac{|G_x|}{|G|} \\ &= \frac{1}{|G|} \sum_{x \in X} |G_x| \\ &= \frac{1}{|G|} \sum_{g \in G} |X_g|. \end{aligned}$$

■

Example of Burnside's Lemma: [Jin] Consider the different colorings of points on a square with 2 colors, up to rotation.



We define a group action G acting on X such that gx is the rotation of some configuration x by some transformation g. Hence, the orbits of this action represent distinct configurations up to rotation. The four transformations $g \in G$ are

(1) *Rotate* $0°$ The number of fixed points X^g given g is a $0°$ rotation is the total amount of possible colors $= 2^4$.

(2) *Rotate* $90°$ The only fixed points would be coloring all vertices the same color $= 2$.

(3) *Rotate* $180°$ The two corresponding vertices of a pair must be the same color, so the number of fixed points is 2^2.

(4) *Rotate* $280°$ Similar to the $90°$ rotation so we have 2 colorings.

Hence, Burnside's Lemma gives the total number of distinct colorings to be

$$|X/G| = \frac{1}{4}(2^4 + 2 + 2^2 + 2) = 6.$$

3. Polya's Enumeration

Polya's Enumeration provides a generalization of Burnside's Lemma, but moreover it allow us to weight "colors". First, we introduce some necessary definitions:

Definition 3.1. Let p be a permutation on X. The **type** of p is the set $\{b_1, b_2, ...b_n\}$ such that b_i is the number of cycles of length i in the cycle decomposition of p.

Definition 3.2. Let the colors $c \in C$ have positive integer weights $w(c)$. The **weight** of a coloring q is the sum of the weights of the colors used:

$$w(q) = \sum_{x \in X} w(q(x)).$$

Definition 3.3. Cycle index Let G be a permutation group. The cyclic index of the group is defined as

$$Z_G(t_1, t_2, ...) = \frac{1}{|G|} \sum_{g \in G} t_1^{b_1} t_2^{b_2} ...,$$

where t_i denotes the cycle length and b_i denotes the number of cycles of length i in the cycle decomposition of g.

Definition 3.4. We also have that the generating function for a set of colors of

$$f(t) = f_0 + f_1 t + f_2 t^2 + ...,$$

where f_i is the number of colors with weight i.

Now that we've covered the prerequisite knowledge of Polya's Enumeration, note that there are two versions of the theorem: namely the unweighted and the weighted versions. We first state the unweighted:

Theorem 3.5. *[ZF]* ***Polya's Enumeration Theorem (Unweighted).*** *Let X be a set with group action induced by a permutation group G on X. Let C be a set of colors on X, and let C^X be the set of functions $f : X \to C$. Then*

$$|C^X/G| = \frac{1}{|G|} \sum_{g \in G} |C|^{c(g)},$$

where c(g) is the number of cycles of g on X.

Proof. This is equivalent to Burnside's Lemma because $|C|^{c(g)}$ is the number of points fixed by g. A point is "fixed" if all elements in the cycle has the same color, which is exactly what Burnside's Lemma states. ■

We now state the weighted version:

Theorem 3.6. *[ZF]* ***Polya's Enumeration Theorem (Weighted).*** *The generating function of the number of colored arrangements by weight is given by*

$$F(t) = Z_G(f(t), f(t^2)...).$$

Proof. It can be shown that

$$\sum_{\text{colorings fixed by g}} t^{w(q)} = \prod_i f(t^i)^{m_i(g)}.$$

We can then show that

$$\frac{1}{|G|}\sum_{g\in G}\prod_i f(t^i)^{m_i(g)} = F(t) = Z_G(f(t), f(t^2), ...).$$

Applying Burnside's on the set of colorings of weight i and summing for all i gives us the desired result.

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Now let's put everything together to see how cycle index, generating functions, and Polya's Enumeration Theorem work together:

Example. Count the number of graphs with 4 vertices.

We want the cycle index of S_4, because the permutation group of the graph is S_4. We count the different cycle lengths:

(1) *6 edges of length 1* This is simply the permutation group (), giving us 1 possibility.

(2) *2 edges of length 1, 2 cycles of length 2* We have 3 different cycles of the form (1x), and $\binom{4}{2} = 6$ possibilities of the form (1a)(bc), giving us 9 possibilities.

(3) *2 cycles of length 3* There are $\binom{4}{3} = 4$ ways to choose the elements in the cycle, and 2 different cycles we can form from any 3 elements, giving us 8 possibilities.

(4) *1 cycle of length 2, 1 cycle of length 4* If the cycle takes the form (1abc), any permutation of the remaining 3 give us a new cycle, giving us $3! = 6$ possibilities.

Hence, we have the cycle index

$$Z_{S_4}(t_1, t_2, t_3, t_4) = \frac{1}{24}(t_1^6 + 9t_1^2t_2^2 + 8t_3^2 + 6t_2t_4).$$

From Definition 3.4, we have the generating function of the graph is

$$f(t) = 1 + t$$

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From Polya's Enumeration Theorem, we have

$$F(t) = Z_G(1 + t, 1 + t^2, 1 + t^3, 1 + t^4)$$

$$= \frac{1}{24}((1 + t)^6 + 9(1 + t)^2(1 + t^2)^2 + 8(1 + t^3)^2 + 6(1 + t^2)(1 + t^4))$$

$$= t^6 + t^5 + 2t^4 + 3t^3 + 2t^2 + t + 1.$$

Hence, we have 1 graph with 6 edges, 1 graph with 5 edges, 2 graphs with 4 edges, 3 graphs with 3 edges, 2 graphs with 2 edges, 1 graph with 1 edge, and 1 graph with 0 edges.

References

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[ZF] Sebastian Zhu and Vincent Fan. Polya enumeration theorem. *MIT*.
[Zha] Alec Zhang. Polya's enumeration. *University of Chicago*.