POLYA'S ENUMERATION THEOREM

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ABSTRACT. In this paper we shall first discuss basic group theory and Burnside's lemma. After that we shall move onto the main topic, Polya's Enumeration Theorem, and its proof.

1. Basic Group Theory

To prove Polya's Enumeration Theorem, we must start with the basics of group theory.

Definition 1.1. A group is a set G, along with an operation $\cdot$ such that it satisfies the following properties:

(1) The operation is associative: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
(2) Given any two elements $g, h \in G$, we have $g \cdot h \in G$.
(3) There exists an element $e \in G$, such that for any $g \in G$, we have

$$e \cdot g = g \cdot e = g.$$

(4) For any element $g \in G$, there exists some element $g^{-1} \in G$ so that

$$g \cdot g^{-1} = g^{-1} \cdot g = e$$

Remark 1.2. A group doesn't have to be commutative, i.e., for any two elements $g, h \in G$, $g \cdot h = h \cdot g$. Commutative groups are also called *abelian groups*, and not commutative groups are called *non-abelian*.

We will now move on to subgroups:

Definition 1.3. Let G be a group, and let H be a nonempty subset of G. We say H is a subgroup of G and we write $H \leq G$ if it satisfies the following properties:

(1) For any $g, h \in H$, we have $g \cdot h \in H$.
(2) For any $h \in H$, $h^{-1} \in H$

In other words, H is a group by itself.

2. Definitions and Lemmas

We will need some more advanced definitions for some theorems.

Definition 2.1. Given a group action ϕ of a group G acting on a set H, we define the *orbit* of an element $h \in H$ to be

$$orb(h) = O_h = \{gh : g \in G\}$$

We will now define the stabilizer and the transformer:

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Definition 2.2. Given a group action ϕ of a group G acting on a set H, we define the *stabilizer* of an element $h \in H$ to be

$$stab(h) = S_{hh} = \{g \in G : gh = h\}$$

Definition 2.3. Given a group action ϕ of a group G acting on a set H, we define the *transformer* of two elements $h, i \in H$ to be

$$trans(h, i) = S_{hi} = \{g \in G : gh = i\}$$

We will also need the definition of a quotient:

Definition 2.4. Given a group action ϕ of a group G on a set X, the *quotient* of ϕ is defined as

$$X/G = \{O_x : x \in X\}$$

Let's start proving some propositions with our new tools!

Proposition 2.5. *For any group action phi of a group G acting on a set X, $S_{xx} \leq G$ for all $x \in X$.*

Proof. Since associativity is "inherited" from the structure of G, we will only have to check for the closure, identity, and inverse properties of a subgroup. For $g_i, g_j \in S_{xx}$ and $x \in X$, we have

- Closure: Obviously, $g_i(g_jx) = g_ix = x$. By the compatibility property of ϕ, we have $(g_ig_j)x = x \Rightarrow g_ig_j \in S_{xx}$.
- Identity: Clearly, the identity element $e \in G$ is in S_{xx} because $ex = x$.
- Inverse: Consider some $g_i \in S_{xx}$. Since $g_ix = x$, we also have $g_i^{-1}(g_ix) = g_i^{-1}x \Rightarrow g_i^{-1}x = (g_i^{-1}g_i)x = ex = x$. Then, by compatibility of ϕ we have $g_i^{-1} \in S_{xx}$

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3. Supporting Theorems and Prerequisites

We will need these supporting theorems and definitions to prove it, especially the Orbit-Stabilizer theorem:

Theorem 3.1. ***Orbit-Stabilizer Theorem***. *Given any group action ϕ of a group G on a set X, for all $x \in X$,*

$$|G| = |S_{xx}||O_x|.$$

We will now prove Burnside's Lemma, which is essential to the proof of the main theorem. We can calculate the order of the group, the size of the stabilizer, and the size of the orbit. But we still can't find the *number* of orbits. This is what Burnside's Lemma, which is attributed to Cauchy is about this:

Theorem 3.2. *Burnside's Lemma.* *Given a finite group G, a finite set X, and a group action ϕ of G acting on X, the number of distinct orbits is*

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|,$$

where X^g denotes the set of all elements fixed by g.

Proof. We can see that

$$\sum_{g\in G}|X^g| = |(g,x)\in(G,X) : gx = x| = \sum_{x\in X} S_{xx}.$$

This means that now we only have to prove that

$$|X/G| = \frac{1}{|G|}\sum_{x\in X}|S_{xx}|.$$

Using the Orbit-Stabilizer theorem, we have $|S_{xx}| = \frac{|G|}{|O_x|}$, so we get

$$\frac{1}{|G|}\sum_{x\in X}|S_{xx}| = \frac{1}{|G|}\sum_{x\in X}\frac{|G|}{|O_x|} = \sum_{x\in X}\frac{1}{|O_x|}$$

Since orbits partition X, we can split up X into disjoint orbits of X/G. This means we can rewrite our sum:

$$\sum_{x\in X}\frac{1}{|O_x|} = \sum_{A\in X/G}\sum_{x\in A}\frac{1}{|A|} = \sum_{A\in X/G} 1 = |X/G|,$$

where A is an orbit in X. Thus, $|X/G| = \frac{1}{|G|}\sum_{x\in X}|X^g|$, so we are done. ∎

Polya's Enumeration Theorem involves multiple definitions as it uses functions from one finite group to another.

Definition 3.3. Type. Let p be a permutation on X. Then, we define the *type* of p to be the set $\{b_1, b_2, \dots, b_n\}$, where b_i is the number of cycles of length i in the cycle decomposition of p.

Definition 3.4. Cycle Index Polynomial We define the *Cycle Index polynomial* Z_ϕ of the group action ϕ is

$$Z_\phi(x_1,\dots,x_n) = \frac{1}{|G|}\sum_{g\in G}\prod_{i=1}^{n} x_i^{b_i(g)},$$

where $b_i(g)$is the ith element of the type of the implied permutation $p_g \in Sym(X)$.

Definition 3.5. Function equivelence. Two functions $f \in Y^X$ are said to be equivalent under the action of $G(f_1 \sim_\phi f_2)$ if they are in the same orbit of ϕ', i.e., there is a $g \in G$ so that $f_2 = gf_1$.

Definition 3.6. Configuration. A *configuration* is an equivalence class of the equivalence relation $\sim_G$ on Y^X.

Definition 3.7. Weight. Let $w : Y \to \mathbb{R}$ be an assignment to each element in Y. We define the *weight* of a function $f \in Y^X$ to be

$$W(f) = \prod_{x \in X} w(f(x)).$$

One last definition:

Definition 3.8. Configuration Generating Function (CGF) Let C be the set of all configurations c. Then the **CGF** is

$$F(C) = \sum_{c \in C} W(c).$$

4. Polya's Enumeration Theorem

We can now prove Polya's Enumeration Theorem, both its weighted and unweighted forms. The unweighted form comes very easily from Burnside's Lemma.

Theorem 4.1. ***Polya's Enumeration Theorem (Unweighted).*** *Let G be a group and X, Y be finite sets, where $|X| = n$. Then for any group action ϕ of G on X, the number of distinct configurations in Y^X is*

$$|C| = \frac{1}{|G|} \sum_{g \in G} |(Y)|^{c(g)},$$

where $c(g)$ denotes the number of cycles in the cycle decomposition of $p_g \in Sym(X)$, the permutation of X associated with the action of g on X.

Proof. Because configurations are orbits of ϕ', we can see that $|C| = |Y^X/G|$ under ϕ'. We can apply Burnside's Lemma to the finite set Y^X with group action ϕ'. Then, we have

$$|Y^X/G| = \frac{1}{|G|} \sum_{g \in G} |(Y^X)^g|.$$

Now, all we have to show is that $|(Y^X)^g| = |Y|^{c(g)}$. A function $f \in Y^X$ will remain constant under the action of g if and only if all elements in X in each cycle are assigned the same set element in Y. This means that there are $|Y|$ choices of elements in Y for each of the $c(g)$ cycles in the cycle decomposition, and we are done. ■

Let's move onto the weighted version:

Theorem 4.2. ***Polya's Enumeration Theorem (Weighted).*** *Let G be a group and X, Y be finite sets, where $|X| = n$. Let w be a weight function on Y. Then for any group action ϕ of G on X, the CGF is given by*

$$Z_\phi\left(\sum_{y \in y} w(y), \sum_{y \in Y} w(y)^2, \ldots, \sum_{y \in Y} w(y)^n\right).$$

Proof. To prove this, we will need a simple lemma:

Lemma 4.3.

$$|C| = \frac{1}{|G|} \sum_{g \in G} |\{f \in Y^X | (\forall x \in X)(f(gx) = f(x))\}|$$

Proof. Let ϕ'_R be the group action on Y^X induced by ϕ:

$$\phi'_R : (f, g) \to f'_R = f \circ p_g = \{(x, f(\phi(g, x)))|x \in X\},$$

where $f \in Y^X$ and $g \in G$. To complete the proof, we apply Burnside's Lemma, and we are done. ■

We can now set ϕ'_R as our group action on Y^X. We let $A(w) = \{c \in C|W(c) = w\}$ which is the set of all configurations with weight w. We then have $S_{gg} = \{f \in Y^X|f = fg\}$ is the set of all functions stabilized by g. We then let $S_{gg}(w) = \{f \in Y^X|f = fg, W(f) = w\}$ be the set of all functions that stabilize g with weight w. Then, we have

$$|A(w)| = \frac{1}{|G|}\sum_{g\in G}|S_{gg}(w)|.$$

We group of CCF by weights, but since our sum is finite, we switch the order:

$$CGF = \sum_{c\in C} W(C) = \sum_{w} w|S_{gg}(w)| = \frac{1}{|G|}\sum_{g\in G}\sum_{f\in S_{gg}} W(f).$$

Since G permutes X through the group action, we have that the corresponding permutation p_g for $g \in G$ has a cycle decomposition $C_1, \ldots, C_k$, where $k \leq n$. It follows that if $f \in S_{gg}$, we have $f(x) = f(gx) = f(g^2x) = \ldots$ for all $x \in X$, $g \in G$, and f is constant on each cycle C_i in the cycle decomposition. Now, we have

$$\sum_{f\in S_{gg}} W(f) = \sum_{f\in S_{gg}}\prod_{x\in X} w(f(x)) = \sum_{f\in S_{gg}}\prod_{i=1}^{k}\prod_{x\in C_i} w(f(x)) = \sum_{f\in S_{gg}}\prod_{i=1}^{k} w\left(f\left(x_i\right)\right)^{|C_i|},$$

where $x_1 \in C_i$. Now, we let $|Y| = m$. Since we are summing over all $f \in S_{gg}$, we will have to cover all possible assignments of $y \in Y$ to cycles C_i. Thus, so our equation becomes

$$\sum_{f\in S_{gg}} W(f) = \prod_{i=1}^{k}\Big(w\left(y_1\right)^{|C_i|} + \ldots + w\left(y_m\right)\Big)^{|C_i|} = \prod_{i=1}^{k}\sum_{y\in Y} w(y)^{|C_i|}.$$

We plug it into our GCF expression to get

$$CGF = \frac{1}{|G|}\sum_{g\in G}\left(\prod_{i=1}^{k}\sum_{y\in Y} w(y)^{|C_i|}\right).$$

Regardless of cycle length, by definition of the type, there will be $b_j(g)$ cycles of length j, so our expression becomes

$$CGF = \frac{1}{|G|}\sum_{g\in G}\prod_{j=1}^{n}\left(\sum_{y\in Y} w(y)^j\right)^{b_j(g)} = Z_\phi\left(\sum_{y\in Y} w(y), \sum_{y\in Y} w(y)^2, \ldots, \sum_{y\in Y} w(y)^n\right),$$

and we are done.

■

Polya's Enumeration Theorem has many applications in places where you would not expect it to come up in, such as organic chemistry. For applications of Polya's Enumeration Theorem, refer to [Noe10].

References

[Noe10] Amanda Noel. Counting and coloring with symmetry: A presentation of polya's enumeration theorem with applications. *Home*, Sep 2010.