An Introduction to Category Theory

Sanjay Gollapudi

June 2020

1 Introduction

This paper is going to talk about Categories or Category Theory. Category Theory is the way to classify collections of certain objects and the ways they map to each other through mappings called *morphisms*. Some examples of Categories can be Groups, Sets, Vector Spaces, etc and they each have their own respective morphisms.

Definition 1. *For a set of Objects to be considered a category they have to follow 3 properties:*

1. ***Morphisms**: For every pair of Objects in a Category, there exists a set of morphisms that maps one object to the other. This can be notated by **HOM**(Obj$_1$,Obj$_2$). If there is a specific morphism you want to represent, lets say called f, then we notate the morphism by **f** as $f : Obj_1 \to Obj_2$.*

2. ***Identity**: For every Object, **Obj**, there is some identity morphism e_{Obj} such that every Obj maps to itself or **HOM**(Obj,Obj) or $f : Obj \to Obj$.*

3. ***Composition**: For every 3 Objects in C, there exists a composition from **Hom**(Obj$_1$,Obj$_2$)×**Hom**(Obj$_2$,Obj$_3$) is **HOM**(Obj$_1$,Obj$_3$). If $\boldsymbol{f} : Obj_1 \to Obj_2$ and $\boldsymbol{g} : Obj_2 \to Obj_3$, then $(g \circ f) : Obj_1 \to Obj_3$ is the composition of **f** and **g**.*

Example. *An example of category is the Group. A group is a category whose objects are groups, which we have covered extensively in class. The morphism that maps one group to other is a group homomorphism, which we have also already learned in class.*

Definition 2. *Relationships between different categories are called Functors ad relationships between these Functors are called Natural Transformations. A Covariant Functor between categories C_1 and C_2, denoted by $F : C_1 \to C_2$, consists of:*

- *for every object c in C_1, there exists an $F(c)$ in C_2*
- *for every morphism $f : c_1 \to c_2$ there is a morphism $F(f) : F(c_1) \to F(c_2)$*

and the Functor must satisfy two properties

- $F(f \circ g) = F(f) \circ F(g)$ *where* $f : c_1 \to c_2$ *and* $g : c_2 \to c_3$
- $F(e_c) = e_{F(c)}$ *where* e_c *is the identity morphism.*

An isomorphism $F : C_1 \to C_2$ *is defined to be a functor* F *from* C_1 *to* C_2 *which is a bijection on the set of arrows and objects. Another way to check if a functor is isomorphic: a functor* $F : C_1 \to C_2$ *is isomorphic if and only if there is a functor* $G : C_2 \to C_1$ *where* $F \circ G$ *and* $G \circ F$ *equal the identity.*

Other important terms to note: A functor $F : C_1 \to C_2$ *is full if for every pair* c, c' *of objects in* C_1 *and every arrow* $g : Fc \to Fc'$ *of* C_2*, there is an arrow* $f : c \to c'$ *of* C_1 *such that* $g = Ff$*.*

A functor $F : C_1 \to C_2$ *is considered faithful when every pair* c, c' *of objects in* C_1 *and for every pair of parallel arrows* $f_1, f_2 : c \to c'$*, the quality* $Ff_1 = Ff_2 : c \to c'$ *would mean that* $f_1 = f_2$

Definition 3. *As explained earlier, one can consider the relationships between functors as Natural Transformations:*

- *Given two functors* $F, G : C_1 \to C_2$ *a natural transformation* $\tau : F \to G$ *is a function which assigns each object* $c \in C_1$ *an arrow* $\tau_c = \tau c : Fc \to Gc$ *of* C_2 *so that every arrow* $f : c \to c'$ *in* C *creates a commutative diagram as shown below.*

$$\begin{array}{ccccc} c & \quad & Fc & \xrightarrow{\tau c} & Gc \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle Gf} \\ c' & & Fc' & \xrightarrow{\tau c'} & Gc' \end{array}$$

A natural transformation is also called a morphism of functors. If a natural transformation τ *has an inverse for every* τc *in* C_2 *then it is called a natural equivalence or* $\tau : F \cong G$

Now we branch off into hom-sets.

Definition 4. *Given two objects* c_1, c_2 *in a category* C *the hom-set is defined as:*

$$hom_c(c_1, c_2) = \{f \mid f \text{ is an arrow s.t. } f : c_1 \to c_2 \in C\}$$

consists of all arrows of the category with domain c_1 *and codomain* c_2*. A category can be defined using these hom-sets in the following properties:*

1. *a set of objects* $c_1, c_2, c_3, \cdot$
2. *a function which takes each ordered pair* (c_1, c_2) *and applies a hom-set* $hom(c_1, c_2)$

3. *for each ordered triple* $< c_1, c_2, c_3 >$ *there exists a function:*

$$hom(c_1, c_2) \times hom(c_2, c_3) \to hom(c_1, c_3)$$

This is called composition and is denoted by $< g, f > \to g \circ f$ *for* $g \in hom(c_2, c_3)$ *and* $f \in hom(c_1, c_2)$*.*

4. *For each object c, there is an element* $id_c \in hom(c, c)$ *which is called the identity element of c.*

5. *If* $< g, f > \neq < g', f' >$ *then* $hom(g, f) \cap hom(g', f') = \emptyset$

With a couple more definitions we will be able to tackle Yonedas Lemma, but first We need to define small sets and locally small categories.

Definition 5. *A locally small category is a category for which all objects* c_1, c_2 *the hom class* $hom(c_1, c_2)$ *is actualy a set called the hom-set which we defined earlier.*

A small category is a categories whose objects and hom classes are sets and not classes.

Definition 6. *Finally we define the covariant hom functor which is denoted by:* $h^A = Hom(A, -)$*, where A is an object in a locally small category and the functor maps to the category of sets, denoted by* ***Set****.*

This functor takes an object X and maps it to $Hom(A, X)$ *and takes a morphism* $f : X \to Y$ *and maps it to* $f \circ -$ *which would then map a morphism* $g \in Hom(A, X)$ *to* $f \circ g$*. In other words we have the following:*

1. $h^A(X) \to Hom(A, X)$

2. $h^A(f) = hom(A, f)$ *and* $h^A(f)(g) = f \circ g$

Now we have everything we need to being Yoneda's Lemma!

Lemma 1. *(Yoneda's Lemma) Let F be an arbitrary functor from* $\mathcal{C}$ *to* ***Set****. For each object* $A \in \mathcal{C}$*, the natural transformations* $Nat(h^A, F) \equiv Hom(Hom(A, -), F)$ *from* h^A *to F are in a one to one correspondence with F(A) or*

$$Hom(Hom(A, -), F) \cong F(A)$$

Proof. A natural transformation η from h^A to F has a corresponding element x in F(A) s.t. x = $\eta_A(id_A)$. This is the case as $\eta_A : Hom(A, A) \to F(A)$ which maps a morphism from A to A to an element in F(A). Since the morphism id_A lies in Hom(A,A) then the morphism id_A is mapped to an element in F(A) associated to η. This element corresponds to the image of id_A under the morphism η_A.
Similarly a given element a in F(A) has a corresponding natural transformation

$\eta(f) = F(f)(x)$. Take an element $a \in F(A)$ which is then mapped to a natural transformation

$$\eta_a : h_A \to F$$

whose morphism on an object X

$$\eta_a(X) : Hom(A, X) \to F(X)$$

can then be written as

$$\eta_a(X)(f) = F(f)(a)$$

where f is a morphism between any 2 objects $f : X \to Y$. Since F is a functor we have the following diagram

$$\begin{array}{ccc} h_A(X) & \xrightarrow{\eta_X} & F(X) \\ \Big\downarrow{\scriptstyle h_a f} & & \Big\downarrow{\scriptstyle F f} \\ h_A(Y) & \xrightarrow{\eta_Y} & F(Y) \end{array}$$

which obviosuly commutes so therefore we have found a natural transformation η. Combining both of these steps we can see there is a bijection between Nat(h_A, F) and $F(A)$ through the map $\eta \to \eta_A id_A$.

□

Another special case of Yoneda's lemma we have to consider is when the second functor F is actually a hom-set functor lets say h_B. Basically the new statement becomes

Theorem 2. *Nat*$(h_A, h_B) \cong Hom(B, A)$. *This implies natural transformations between hom functors are bijective with morphisms of the respective objects. Given a morphism* $f : B \to A$ *the associated natural transformation is denoted by* $Hom(f, -)$.

Through the respective mappings of objects and morphism to their functors and natural transformations we can define a new covariant functor called h^- which can be described as $h^- : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}}$ where $\mathbf{Set}^{\mathcal{C}}$ denotes the functor category which is a category whose objects are functors from $\mathcal{C}$ to **Set** As a result, this h^- is actually fully faithful and can draw the conclusion that the category $\mathcal{C}$ is really isomorphic or has a bijection to $\{h_A \mid A \in \mathcal{C}\}$.

Thus ends my expository paper on Category theory. I started off with basic definitions and then built more and more definitions until the reader had enough material to understand the proof to Yoneda's Lemma.

References

[1] Saunders Mac Lane. *Categories for the Working Mathematician.* University of Chicago, Chicago, Illinois, 1991.

[2] Adam Topaz. *The Yoneda Lemma.* University of California: Berkeley, Berkeley, California.

[3] Emily Riehl. *Category Theory in Context.* Cambridge University Press, Cambridge, England.