Rubik's Cube Groups

Samrit Grover and Ayush Aggarwal

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1 Cube Notation

A Rubik's cube is made up of 27 small cubes which are typically referred to as cubies. There are 8 corner cubies, 12 edge cubies, and 6 center cubies.

The names of the six faces are right(r), left(l), front(f), back(b), up(u), and down(d). To identify corner cubies we can list the faces they are visible from in clockwise order. For example the cubie in the upper, right, front corner is called urf. For edge cubies, we list the 2 faces they are visible from and for a center cubie we just list the face it is on (e.g. f). Cubicles will refer to the positions that cubies can go in. When a cube is rotated, the cubies move, but the cubicles don't. Finally we need to define names for the rotations or moves. These will be R, L, U, F, D, B and they denote clockwise twists of their corresponding face. Similarly R' is a counterclockwise rotation of the right face. Finally the notation R^2 means to rotate the right face clockwise twice.

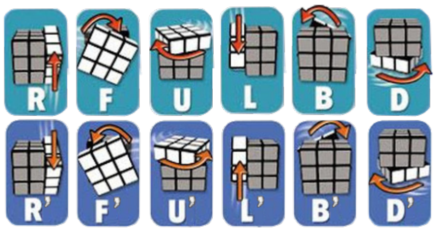

Figure 1: Notations of a Rubik's Cube

2 The Rubik's Cube as a Group

We can define G as the group of all possible moves of a Rubik's Cube where two moves are considered the same if they result in the same orientation. For example, a 180 degree rotation clockwise is the same as 180 degree rotation counterclockwise. The group operation is defined as such: If M_1 and M_2 are two moves then $M_1 * M_2$ is the move where you do M_1 first then M_2.

G is definitely closed as if M_1 and M_2 are moves then so is $M_1 * M_2$ as G contains all possible moves of a Rubik's cube and $M_1 * M_2$ will be one of these.

The identity element is to simply do nothing. This can be achieved by something like rotating a face 4 times which doesn't change the configuration.

If M is a move we can reverse all the steps of M to get M', the inverse of M. A Rubik's cube technically can't use the principle of associativity as Rubik's cube moves are just an order. Saying $F(RU)$ would normally mean do RU first and then F, but this concept doesn't apply to Rubik's cube as the way to express RU first and then F is just RUF

3 Subgroups of the Cube

Subgroups of a Rubik's cube can be generated many things. Some examples are F which generates the subgroup F, F^2, F^3, F^4. In fact we conjecture that any move P generates a subgroup of G. If the cube starts in a solved state, then if a move P is performed successively it will return to its original solves state.

Proof: Let P be any cube move sequence. After some number of times n that P is applied it cycles back to the same arrangement k, where $k < n$ and n is the soonest an arrangement appears for a second time. Then we know $P^k = P^n$. So if we show that $k = 0$ then we show that the cube has returned to its original state. If $k = 0$ the we are already done, so we will prove by contradiction that k must be 0. If $k > 0$ then we apply P^{-1} to both P^k and P^n. Since $P^k = P^n$ we know that $P^k P^{-1} = P^n P^{-1} \rightarrow P^{k-1} = P^{n-1}$.But this is contradictory, since we said that n is the first time that arrangements repeat, so therefore k must equal 0 and every move sequence eventually cycles through the initial state again first before repeating other arrangements.

4 Macros

We can develop different macros or cube combinations to help us solve the cube.

4.1 Commutator

It is fairly obvious that moves aren't commutative in a Rubik's cube so we can describe their relative commutativity with a commutator. If P and M are two cube moves their commutator is $PMP^{-1}M^{-1}$.

Let the support of an operator be all the cubies changed by it. Then two operations are commutative if either they are the same operation or if $\mathit{support}(P) \cap \mathit{support}(M) = \emptyset$, that is, if each move affects completely different sets of cubies.

If the commutator isn't the identity we can look at their relative commutativity by the number of cubies changed by applying the commutator. Looking at the intersections of the supports can tell us how useful certain commutators are. Useful moves have only a small number of cubies affected as these moves allow us to change parts of the cube while leaving the portions we already solved intact.

Some useful commutators are as follows:

- $FUDLLUUDDRU$ flips exactly one edge cubie on the top face,
- $R'DRFDf$ twists one cubie on a face, and
- $R'DR$ cycles three corners

Let's take a look at two specific moves to see how making them into a commutator will help. Our first move X will be $F'DFLDL'$. This move rotates the front top left corner clockwise without disturbing the rest of the top layer. However this does mess up the bottom two layers of the cube. Our second move will simply be $Y = U$. This moves the front top right corner into the top left position. After we do the move XY to our solved cube we will get something that looks like this.

Figure 2: Cube after move XY

The top left corner (when viewed from the orange side) has been flipped while the rest of the cube is messed up. To fix this we can use a commutator. The commutator for XY is $XYX^{-1}Y^{-1}$, so we need to do the move $X^{-1}Y^{-1}$. This move is then $(LD'L'F'D'F)U'$, which repairs the damage to the bottom two layers while simultaneously flipping another corner. After this move our cube looks like this.

Figure 3: Cube after move $XYX^{-1}Y^{-1}$

The whole cube is intact except for the two corners that we wanted to rotate clockwise. Commutators like these allow us to change the parts of the cube we haven't solved while keeping the solved portions the way they already are.

Theorem: If X and Y only affect one cubie in common, and if no other cubie is affected by both X and Y, then $XYX^{-1}Y^{-1}$ is a three-cycle. i.e. This means that there are cubies a, b, c such that $XYX^{-1}Y^{-1}$ moves a to b, moves b to c, moves c to a, and does not move anything else.

Proof: Let a be the cubie that is moved by both X and Y. Then b is the cubie that Y moves to a, which means Y^{-1} moves a to b. Similarly c is the cubie that X moves to a, and so X^{-1} moves a to c. This shows that the moves X and Y and their inverses can only affect these three cubies and cycle them.

4.2 Conjugation

Let N be some macro that performs a cube operation. Then we say for some cube move P, PNP^{-1} is the conjugation of M by P. Conjugating a group element is another very useful tool that will help us describe and build useful macros. First we will introduce a couple useful definitions: An equivalence relation is any relation $\sim$ between elements that are:

- Reflexive: $x \sim x$
- Symmetric: If $x \sim y$ then $y \sim x$
- Transitive: If $x \sim y$ and $y \sim z$ then $x \sim z$

We will let the relation $\sim$ be conjugacy. So if for some $g \in G$, $x \sim y$, then $gxg^{-1} = y$. The following proves conjugacy is an equivalence relation:

- Reflexive: $gxg^{-1} = x$ if $g = 1$, so $x \sim x$
- Symmetric: If $x \sim y$, then $gxg^{-1} = y$, so multiplying each side by g on the right and g^{-1} on the left gives $x = g^{-1}yg$
- Transitive: If $x \sim y$ and $y \sim z$, then $y = gxg^{-1}$ and $z = hyh^{-1}$, so $z = hgxg^{-1}h^{-1} = (hg)x(hg)^{-1}$, so $x \sim z$

Just like commutators conjugations can also be used to move certain cubies while keeping the rest of the cube the same as before. Let's say we wanted to cycle three corners on the top face. We can do this by using already known moves and making conjugations out of them along with using commutators. Our moves will be $X = FLF', Y = R^2, Z = F^2$. Both X and Y only move the front bottom right cubie so the commutator $XYX^{-1}Y^{-1}$ is a three-cycle of cubies. If we just use this commutator our cube looks like this:

Figure 4: Cube after commutator $XYX^{-1}Y^{-1}$

Three corners have been switched and the rest of the cube has stayed the same. However, these corners are far apart and so cycling these three corners won't really be helpful. While commutators change certain things and those things alone, that doesn't necessarily mean they are helpful. Instead we can make a conjugate with the move $Z = F^2$. This simple move moves two of the corners we switched here into the top layer. When we make a conjugate with Z and our commutator we get this cube:

Figure 5: Cube after conjugate $Z(XYX^{-1}Y^{-1})Z^{-1}$

This move now cycles three corners all on the top layer. It also keeps all the yellow stickers pointing up which makes this conjugate very useful for solving the top layer when the cubies are arranged wrong, but facing the correct way. It also shows the usefulness of conjugates and how they can be used to convert useless commutators into useful ones and it can even make normal moves useful as well.

Macros such as conjugates and commutators allow you to take essentially any move and use it so that it doesn't change the rest of the cube. Even if you don't know the normal algorithms for solving a cube you can intuitively make your own using commutators and conjugates.

4.3 Impossible Moves

While there are so many moves possible in a Rubik's Cube, there are three positions that can't be reached while keeping the rest of the cube the same. They are:

- Flipping a single edge
- Permuting two cubies
- Rotating a corner.

It is a fairly well known fact that there are about 43 quintillion combinations of Rubik's cube, and the reason there aren't more is because of these impossible moves. If we overcount the possible combinations of a cube we will get that there are 8 corner pieces which can be arranged in 8! ways and each has three orientations for 3^8 total orientations. Similarly for edges we get 12! arrangements and 2^12 orientations. If we multiply these numbers together we get about 519 quintillion combinations of the Rubik's Cube. However we have to divide this by 12 as there are 2 ways to orient an edge, 2 ways to permute two cubies, and 3 ways to rotate a corner, all of which are impossible moves.

5 Solving the Cube

There are many ways to solve Rubik's Cube but we'll show the easiest one.

5.1 The White Cross

It doesn't really matter what color face we decide to start with,but for now let's solve the white face. There are no algorithms for making a white cross on the white face as it can be done very easily and intuitively, especially since you don't have to worry about messing up past progress.

5.2 The White Face

Once you have a white cross find a white corner piece and bring it to either the corner it's supposed to be in or the corner below where its supposed to be (its okay if it isn't oriented properly at this point, it just needs to be in one of those cubicles). From here hold the cube so that the corner cubie you want to orient correctly is on the right and the white face is on the top. Then repeat the commutator $R'D'RD$ until the piece is oriented correctly. Do the same thing for the other 4 corners until the white face is completely solved. Your cube should look like this:

Figure 6: Cube with white face solved

5.3 The Second Layer

Turn your cube upside down so the white layer is now on the bottom. We now need to orient the 4 edges blue-orange, orange-green, green-red, and red-blue. Find one of these edges that is already in the top row (If by some chance all of these edges are in the second layer do any of the two algorithms mentioned in this step). Move that edge piece so that one of its colors lines up with the color of a center cubie. An example would look like this where the orange edge piece lines up with the orange center cubie.

Figure 7: Cube with orange cubies lining up

After this the edge cubie can either go to the right or to the left. In our example the orange-blue cubie needs to go to the right to be put in the orange-blue cubicle. Hold the cube with white on the bottom and the edge cubie you wish to move in front of you. Use the following macros depending on whether the cubie goes to the right or left.

Right: $U'L'ULUFU'F'$

Left: $URU'R'U'F'UF$

5.4 The Yellow Cross

Hold the cube with the yellow face on top and see the pattern of yellow edges on top. The pattern on top will tell you the number of times you need to apply this commutator/conjugate: $FRUR'U'F'$.

If you see a full cross you are done and you don't need to apply the macro.

If you see a horizontal line (if vertical rotate so the line is horizontal) apply the macro 1 time.

If you see a backwards L (again rotate if needed) apply the macro 2 times.

If you see only the center piece and one or no other yellow edges on top apply the macro 3 times.

When you are done your cube should something like this:

Figure 8: Cube with yellow cross on top

5.5 The Yellow Edges

Once you have the yellow cross you need to make sure the other color on the yellow edges line up with their faces. To do this we use the macro $RUR'URU^2R'U$. This macro switches the front and left yellow edges. You can use this macro repeatedly and s little bit of logic to get the edges in the right place.

5.6 The Yellow Corners

Only the yellow corners are unsolved now. For this step we won't worry about orientation, just about getting the yellow corners in the right place. Find a corner already in the right place and hold the cube so that this corner is in the top right corner (if there are none do the macro once and look again) and repeat this macro until they are all in the right place: $URU'L'UR'U'L$. Your cube should look something like this:

Figure 9: Cube with yellow corners in right places

5.7 Orienting the Corners

For the final step hold the cube so that an unsolved corner is in the top right corner. Then repeat the commutator $R'D'RD$ either 2 or 4 times until the corner is properly oriented. The rest of the cube will seem to be messed up, but don't worry about that for now as it will come back together later on. Once the corner is oriented move the top layer and this layer only to put another unsolved corner into the top right. Repeat this process until you have a solved cube (you may need to twist the top layer once at the end).