Matrix Lie Groups

Saadiq Shaikh

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Abstract

First, we will define and explain the idea of a matrix Lie group and how it is connected to the general linear group. Then we give examples of several matrix Lie groups and describe their structure so the reader has some familiarity with what matrix Lie groups look like. Then we will define tangent spaces, the exponential map, and Lie algebras. We will then prove a series of statements to try and demonstrate the connections between these objects. I try to explain everything so only knowledge of basic abstract algebra, basic calculus concepts, and some linear algebra is required.

1 Introduction

We generally think of groups as describing the interactions of discrete elements. However, the study of groups can also be extended to continuous sets. These are called "Lie Groups." Some of the more interesting Lie groups, including many geometric transformations, can be represented with matrices. These are known as matrix Lie groups, and are the easiest to study because of their concrete representation.

Definition 1. A **Lie group** is, roughly speaking, a group with continuous elements. The group operation is multiplication, which, along with inversion, must be a smooth(infinitely differentiable) map.

Lie groups can be thought of as acting on manifolds in the same way that familiar groups like $\mathbb{Z}/n\mathbb{Z}$ act on sets. For this paper, the formal definition of a manifold will not be necessary; just think of manifolds as continuous sets. An example of a simple lie group is $\mathbb{R}^n$ under vector addition.

2 Matrix Lie Groups

Definition 2. $M_n(\mathbb{C})$ is the set of $n \times n$ matrices with entries in $\mathbb{C}$. Similarly, $M_n(\mathbb{R})$ is the set of $n \times n$ matrices with entries in $\mathbb{R}$.

Definition 3. The **General linear group** of degree n over a field $\mathbb{F}$, denoted $GL_n(\mathbb{F})$ is the group of invertible $n \times n$ matrices with entries in $\mathbb{F}$. For example, $GL_n(\mathbb{C})$ denotes the general linear group of $n \times n$ matrices with entries in $\mathbb{C}$.

Definition 4. A **matrix Lie group** is any topologically closed subgroup of $GL_n(\mathbb{C})$. By topologically closed, we mean that if a sequence of matrices (A_n) in the matrix lie group G converges to a matrix A, then A must be in G or A must not be invertible.

Convergence of a $n \times n$ matrix over $\mathbb{C}$ to a matrix M is the same as convergence in $\mathbb{C}^{n^2}$; that is, the sequences of elements in each position in the matrix must converge to the element in the same position in M. An example of a subgroup of $GL_n(\mathbb{C})$ that is not topologically closed is $GL_2(\mathbb{Q})$. Consider the sequence of matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 + \frac{1}{1!} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 + \frac{1}{1!} + \frac{1}{2!} & 0 \\ 0 & 1 \end{pmatrix}, \ldots$$

All these matrices are elementary and have rational entries, so they are in $GL_2(\mathbb{Q})$. This sequence converges to

$$\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}$$

which is invertible but not in $GL_2(\mathbb{Q})$.

Now let's go over some matrix lie groups to gain some familiarity with them.

Definition 5. The **special linear group**, denoted $SL_n(\mathbb{F})$, is the set of $n \times n$ matrices with entries in the field $\mathbb{F}$ that have determinant 1.

Proposition 1. $SL_n(\mathbb{C})$ and $SL_n(\mathbb{R})$ are matrix Lie groups.

Proof. Because any matrix A is invertible iff $\det(A) \neq 0$, because the determinant is multiplicative, and because $\det(A^{-1}) = \frac{1}{\det(A)}$, these are both subgroups of $GL_n(\mathbb{C})$. Since the determinant of an $n \times n$ matrix is a polynomial in n variables, it is continuous, so any sequence of matrices with determinant 1 converges to a matrix with determinant 1. Additionally, $\mathbb{R}$ and $\mathbb{C}$ are both complete. Then any sequence of matrices in $SL_n(\mathbb{C})$ or $SL_n(\mathbb{R})$ will converge in that same group. Thus, $SL_n(\mathbb{C})$ and $SL_n(\mathbb{R})$ are both matrix Lie groups. $\square$

Now we will introduce the orthogonal group.

Definition 6. The orthogonal group, $O(n)$, is the set of all matrices A with entries in $\mathbb{R}$ with $A \cdot A^T = I$.

Geometrically, the orthogonal group is the set of all transformations that preserve distance in $\mathbb{R}^n$.

Definition 7. The special orthogonal group, $SO(n)$, is a subgroup of $O(n)$, containing all orthogonal matrices over $\mathbb{R}^n$ with determinant 1.

Geometrically, the special orthogonal group consists of transformations that preserve distance as well as orientation. Proving $O(n)$ and $SO(n)$ are matrix Lie groups is a simple exercise and is similar to the proof of proposition 1. Now let's look at the 2-dimensional case of the special orthogonal group.

Definition 8. $SO(2)$ is the group of rotations in the plane; it is a special case of the orthogonal group. $SO(2)$ is a matrix Lie group, and is isomorphic to the unit circle.

Rotating by an angle θ corresponds to left multiplication by the invertible matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

This can be easily verified using trigonometry.

3 Tangent Spaces

Now we will discuss tangent spaces, which are important in describing the relation between matrix Lie groups and Lie algebras. Roughly speaking, tangent spaces are to matrix lie groups as tangent lines are to curves in $\mathbb{R}^2$. To describe tangent spaces, we must first define the notion of a path.

Definition 9. A **path** is a continuous map $\lambda : [0,1] \to M_n(\mathbb{R})$. In this case, it can be thought of as a $n \times n$ matrix where each entry $a_{i,j}(t)$ is a function $f : [0,1] \to \mathbb{R}$.

Definition 10. A path is **smooth** if it is differentiable (i.e. the derivatives of all the $a_{i,j}(t)$ exist).

Definition 11. The **tangent vector** of a smooth path $A(t)$ at t is $A'(t)$. Define an equivalence relation on two smooth paths $A(t)$ and $B(t)$ both passing through point P at $t = 0$ of a matrix lie group G such that $A(t)$ and $B(t)$ are in the same equivalence class if $A'(0) = B'(0)$. The set of all such equivalence classes is the **tangent space at point P**, and is denoted $T_P G$.

The tangent space at point P can be thought of as containing all possible "velocities," or directions to travel to, from point P. In this paper we will only be talking about the tangent space at the identity, $T_1 G$. It turns out that tangent spaces at different points are all isomorphic, so we use $T_1 G$ as it is easier to work with and because every group must have an identity element. From now on, "tangent space" refers to $T_1 G$.

A useful tool in finding the tangent space of a Lie group is the exponential map. The exponential map is the analogue of the exponential function in $\mathbb{R}$ for matrix Lie groups.

Definition 12. The **exponential map** of a matrix A is defined as

$$e^A = I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} \dots.$$

Properties of the matrix exponential:

1. $e^0 = I$
2. $e^{X^T} = (e^X)^T$ (where X^T is the transpose of X)
3. $e^{X^*} = (e^X)*$ (where X^* is the conjugate transpose of X)

Proposition 2. $e^{A+B} = e^A e^B$

This follows from expanding the power series of the left and right side of the equation. We can use the exponential map to find the tangent space of some Lie algebras, because it explicitly gives us a smooth path with a derivative that is easy to compute. Let's start with the general linear group.

Proposition 3. The tangent space of $GL_n(\mathbb{R})$ is $M_n(\mathbb{R})$.

Proof. Consider any matrix $X \in M_n(\mathbb{R})$. Consider the path $A(t) = e^{tX}$. Since $e^{-tX} = A(t)^{-1}$ for all t, this path is in $GL_n(\mathbb{R})$. Then since this path has tangent vector $Xe^0 = X$ at the identity, we have that $M_n(\mathbb{R})$ is a subset of the tangent space. We also know that the tangent space must be a subset of $M_n(\mathbb{R})$ as the image of any path is $M_n(\mathbb{R})$. Then we have that the tangent space of $Gl_n(\mathbb{R})$ is $M_n\mathbb{R})$. ☐

Definition 13. A **skew-symmetric** matrix is a matrix A such that $A + A^T = 0$.

Proposition 4. The tangent space of $O(n)$ is the set of all real skew-symmetric matrices.

Proof. Consider the path $t \to e^{tX}$ for any real skew-symmetric matrix X. Then this path is in $O(n)$ since

$$e^{tX}(e^{tX})^T = e^{tX}etX^T = e^{t(X+X^T)} = I.$$

Then the tangent vector at 0, which is X, is in the tangent space, and X must be skew-symmetric for e^tX to be in $O(n)$ for any T. Thus, $T_1O(n)$ is simply the set of real skew-symmetric matrices. ☐

Proposition 5. The tangent space of $SO(n)$ is the same as the tangent space of $O(n)$.

Proof. To prove this, we show that any smooth path passing through the identity in $O(n)$ is also in $SO(n)$. Let $c(t)$ be such a path in $O(n)$. Then since it passes through the identity, which has determinant 1, every other point on the path must also have determinant 1 as the determinant is continuous and the only possible determinants in $O(n)$ are ± 1. But since all these points have determinant 1 and are in $O(n)$, they are also in $SO(n)$ by its definition. Thus, the set of smooth paths passing through the identity is the same in both groups, so they have the same tangent vectors at the identity. ☐

4 Lie Algebras

Now we will introduce Lie Algebras.

Definition 14. A **Lie Algebra** is a vector space $\mathfrak{g}$ equipped with an operation $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ called the **Lie Bracket**. The operation must satisfy the following:

- It is bilinear.
- $[x,x] = 0$, or equivalently $[x,y] = -[y,x]$ for all $x, y \in \mathfrak{g}$
- $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

The second condition is known as skew-symmetry and the third is the Jacobi identity.

At first glance, the association between Lie Algebras and Lie Groups seems unclear. This is where we bring tangent spaces back into the discussion: it turns out the tangent space of a Lie group is a Lie algebra of the group.

Proposition 6. T_1G, where G is a matrix Lie group, is a vector space.

Proof. We know by the definition of a path that any tangent vector must be a $n \times n$ matrix over $\mathbb{R}$. So $T_1G \subset M_n(\mathbb{R})$, which means we only need to prove that T_1G is closed under addition and scalar multiplication since $M_n(\mathbb{R})$ is a vector space. To show the tangent space is closed under addition, consider any two smooth paths $A(t)$ and $B(t)$ such that $A(0) = B(0) = 1$. Then $A'(0)$ and $B'(0)$ are tangent vectors in T_1G. Consider $C(t) = A(t)B(t)$. Since A and B are smooth, C must also be smooth. Furthermore, $C(0) = A(0)B(0) = 1$, so the tangent vector $C'(0)$ is in T_1G. By the product rule, $C'(0) = A'(0)B(0) + A(0)B'(0) = A'(0) + B'(0)$. Thus, the sum of any two tangent vectors in T_1G is also in T_1G, so T_1G is closed under addition. Also, for any $r \in \mathbb{R}$, $D(t) = A(rt)$ is a smooth path that starts at the identity with $D'(t) = A'(rt) = rA'(t)$ by the chain rule. Thus, the tangent space is closed under scalar multiplication as well, so it is a vector space. □

Proposition 7. $M_n(\mathbb{C})$ is a Lie algebra under the Lie Bracket $[A,B] = AB-BA$ (this is the standard Lie bracket, known as the commutator bracket).

Proof. This is left as an exercise to the reader. The proof just amounts to checking the commutator bracket satisfies the Jacobi Identity and is skew-symmetric. □

Theorem 1. The tangent space equipped with the commutator bracket is a Lie algebra.

Proof. Since $T_1G \subset M_n(\mathbb{C})$, we only need to prove that it is closed under the Lie bracket in proposition 2, and it automatically satisfies the requirements of the Lie bracket (this is similar reasoning as to how we prove a subset of a vector space is a vector subspace).

Consider any two elements X and Y in T_1G. Then there are paths $A(t)$ and $B(t)$ such that $A(0) = B(0) = I$ and $A'(0) = X, B'(0) = Y$. Consider the path $A(t)YA(t)^{-1}$. This is a smooth path so its tangent vector must be in $T_1(G)$. So we have

$$A'(0)YA(0)^{-1} + A(0)Y(-A'(0)) = A'(0)Y - YA'(0) = XY - YX \in T_1(G).$$

Thus, the tangent space is the Lie algebra. □

So the connection between Lie algebras and Lie groups are becoming clearer. Essentially, the the exponential map gives a way of finding the tangent space of a Lie group, which can be turned into a Lie algebra. Looking back at how we found tangent spaces of Lie groups using the exponential map, it seems like the most useful part of the exponential map was that it "converted" multiplication to addition through the laws of exponentiation. Indeed, it seems very telling that the tangent space of matrices whose multiplicative inverses are equal to their transpose consists of matrices whose additive inverses are equal to their transpose. To understand this better, let's define the idea of a one parameter subgroup.

Definition 15. A **one-parameter subgroup** of a Lie group G is a smooth homomorphism $\lambda : (\mathbb{R}, +) \to G$. That is, it defines a homomorphism from the additive group of real numbers to the multiplicative Lie group G. Then we have that λ is a smooth curve with $\lambda(a + b) = \lambda(a)\lambda(b)$.

It is easy to see that the exponential map is a one-parameter subgroup since $e^{A+B} = e^A e^B$. But what makes it special is that, for matrix groups, every one-parameter subgroup is a matrix exponential:

Proposition 8. Every one-parameter subgroup of a matrix group G is of the form $\lambda(t) = e^{tX}$ where X is a matrix that is a tangent vector at the identity.

Proof. Suppose we have a one-parameter subgroup $\lambda(t)$. Then we know that for all $a, b \in \mathbb{R}$,

$$\lambda(a)\lambda(b) = \lambda(a + b).$$

If differentiate with respect to b, we get

$$\lambda(a)\lambda'(b) = \lambda'(a + b)$$

Setting $a = 0$ gives

$$\lambda(a)\lambda'(0) = \lambda'(a).$$

This is a well-known differential equation with unique solution $\lambda(a) = e^{\lambda'(0)a}$. Since $\lambda'(0)$ is clearly a tangent vector at the identity, the proof is complete. □

This illustrates the importance of the exponential map; it allows one to rediscover the structure of a Lie group by providing a homomorphism from the Lie group to its Lie algebra.

5 Conclusion

Throughout this paper, we have explored the connections between Lie groups, their tangent spaces, Lie algebras, and the exponential map. The essence of the connection boils down to the exponential map being a one-parameter subgroup which gives a homomorphism from the Lie group to the Lie algebra.

While this demonstrates the basics of matrix Lie groups and their interesting qualities, there are much deeper results in this area of study. One example is Ado's theorem, a very powerful theorem that demonstrates the importance of matrix Lie algebras.

Theorem 2. (Ado's Theorem) Any finite-dimensional Lie algebra over a field of characteristic zero is isomorphic to a square matrix Lie algebra under the commutator bracket.

"Characteristic zero" means that adding the multiplicative identity to itself any amount of times will never yield the additive identity of the field. For example, every subfield of $\mathbb{C}$ is a field of characteristic zero.

Though the proof of this theorem is outside the scope of this paper, it demonstrates the power of matrix Lie algebras and one of the reasons Lie groups are so important: their Lie algebras can be viewed, concretely, as matrix algebras. This is why Lie groups have numerous applications in different areas like theoretical physics and differential equations, and continue to be studied today.

References

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