Sporadic Groups

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1 Introduction

The sporadic groups are 26 finite simple groups that do not fall into the 3 main classes stated in the classification of finite simple groups. Unlike the first three groups, the sporadic groups do not represent infinite families in each group. The first set of 5 sporadic groups were found in 1861, but other 21 were found between 1965 and 1975.

2 Classification Theorem

Theorem: Classification of Finite Simpler Groups asserts that a finite simple group is one of the following:
1) A cyclic group of prime order.
2) An alternating group A_n for $n >= 5$, the group of even permutations of $1, 2, \ldots n$.
3) A group of Lie type; these are more-or-less matrix groups, and fall into a finite number of families. Some of the families require a dimension and a field order to be given to specify the group (for example, the group PSL(n,q) consists of the nn matrices over the field of order q modulo scalar matrices); the others require just a field order, which is itself sometimes restricted (for example, Sz(q), the Suzuki group, where q must be an odd power of 2).
4) One of twenty-six sporadic groups.

3 Types of Sporadic Groups

The sporadic groups are classified into four groups, Mathieu Groups, Leech Lattice/Conway Groups, Monster Groups and Pariah groups. The following diagram shows the subquotient relationship of these groups, and only 6 groups are maximal in this relationship.

¡Insert Image here¿

3.1 Mathieu Groups

3.1.1 Background

The Mathieu Groups were the first sporadic groups to be discovered. Mathieu introduced M_{12} while investigating multiplicative transitive permutation groups.

3.1.2 Multiplicative Transitive Groups

Definition: A permutation group is a group G whose elements are permutations of a given set M and whose group operation is the composition of permutations in G.
Definition: A permutation group G acting on x points is n-transitive if $a_1, a_2, ..a_n$ and $b_1, b_2, b_3, ..b_n$ are two sets of points such that all a_i and b_i are distinct and there exists $g \in G$ mapping a_i to b_i for i between 1 and n.
The Mathieu groups are the five sporadic simple groups $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$. These five groups are multiplicative transitive groups on 11, 12, 22, 23, or 24 objects respectively.
Remark: M_{11} is 4-transitive, M_{12} is 5-transitive, M_{22} is 3-transitive, M_{23} is 4-transitive, and M_{24} is 5-transitive.

3.1.3 Construction

Definition: If $x < y < z$ with x, y, and z being positive integers, a collection $S_1, S_2, ...S_N$ of different subsets of 1, 2, .. n is a (x, y, z)-Steiner system, denoted S(x, y, z) if it satisfies two properties: for all i —S_i— = m and for every subset T of 1, 2, .. n with $|T| = l$, there is exactly one i. Remark: Mathieu Groups are automorphisms of Steiner systems.
Definition: The Mathieu groups can be defined as followed:
$M_{11} = \{\sigma \in S_{11} : \sigma(S) \in S(4, 5, 11) \text{ for all } S \in S(4, 5, 11)\}$.
$M_{12} = \{\sigma \in S_{12} : \sigma(S) \in S(5, 6, 12) \text{ for all } S \in S(5, 6, 12)\}$.
$M_{22} = \{\sigma \in S_{22} : \sigma(S) \in S(3, 6, 22) \text{ for all } S \in S(3, 6, 22)\}$.
$M_{23} = \{\sigma \in S_{23} : \sigma(S) \in S(4, 7, 23) \text{ for all } S \in S(4, 7, 23)\}$.
$M_{24} = \{\sigma \in S_{24} : \sigma(S) \in S(5, 8, 24) \text{ for all } S \in S(5, 8, 24)\}$.
Theorem: The Mathieu groups are simple.

3.2 Leech/Conway Group

3.2.1 Background

The Conway groups are the three sporadic simple groups Co1, Co2 and Co3 along with the related finite group Co0 introduced by Conway in 1968/1969. This contains three sporadic groups Co1, Co2 and Co3. Co0 is the largest of the Conway groups, and this group is the result of automorphisms of the Leech lattice with respect to addition and inner product. It has order 8,315,553,613,086,720,000 but this is not a simple group. The simple groups are created by different automorphisms on this group as described in the construction section below.

3.2.2 What is a Leech Lattice

The Leech lattice is an even unimodular lattice in 24-dimensional Euclidean space, which is one of the best models for the kissing number problem. It was discovered by John Leech in 1967. A kissing number is defined as the greatest number of non-overlapping unit spheres that can be arranged such that they each touch a common unit sphere in n dimensional space. For a one dimensional space, this number is 2, for two dimensional space it is 6, and for three dimensional space it is 12. The determination becomes complex as the number of dimensions increase, and is unknown for many dimensions. This is known for 24 dimension space (196,560) and known as Leech lattice. This has the following properties 1. It is unimodular; i.e., it can be generated by the columns of a certain 2424 matrix with determinant 1. 2. It is even; i.e., the square of the length of each vector in 24 is an even integer. 3. The length of every non-zero vector in 24 is at least 2.

3.2.3 Construction

Definition: Thesegroups can be defined as followed:
C_1 = Quotient of Co0 by its center. This group has an order of 4,157,776,806,543,360,000
C_2 = Automorphisms of fixing a lattice vector of type 2. This group has an order of 42,305,421,312,000.
C_3 = Automorphisms of fixing a lattice vector of type 3. This group has an order of 495,766,656,000.
MCL = automorphisms of preserving a complex structure.
Suz = automorphisms of with a triangle of type 2-2-3.
HS = automorphisms of with a triangle of type 2-3-3.
$J_2 = automorphismsofpreservingaquaternionicstructure.$

3.3 Monster Group

3.3.1 Background

The Monster Group, also known as the Fischer-Griess monster). The Monster Group contains 20 sporadic groups (including itself) as subquotients. These 20 groups were called the "happy family" by Robert Griess, and the 6 exceptions form the Pariah Group.

3.3.2 Moonshine

The Monster Group serves as one of the two main parts of the Moonshine conjecture by Conway and Norton, which combines discrete and non-discrete math. It is as an unexpected connectino between the monster group M and modular functions (the j function): $J(\tau) = \frac{1}{q} + 196884q + 21493760q^2 + ...$ with $q = e^{2\pi ir}$. It is believed that there is a naturally occurring indefinite dimensional graded representation of M, whose graded dimension is given by coefficients of J, who lower weight pieces decompose into irreducible representations. It was suggested thta becausee the graded dimension is just the graded trace of the identity element the graded traces of nontrivial elements g od M on such representation may be interesting. Conway/Norton computer lower-order terms (McKay-Thompson series T_g. By decomposing a coefficients of J into representations of M, strong evidence was found that a graded representation exists. IT was proved that graded traces for all elements in the centralizer of an involution of M and that the vector space constructed, Moonshine MOdule V, has the structure of a vertex operator algebra, with authomorphism gropu being the monster group.

Monster Module according to the Frenkel - Lepowsky - Meurman construction:
1) "The construction of a lattice vertex operator algebra VL for an even lattice L of rank n. In physical terms, this is the chiral algebra for a bosonic string compactified on a torus Rn/L. It can be described roughly as the tensor product of the group ring of L with the oscillator representation in n dimensions (which is itself isomorphic to a polynomial ring in countably infinitely many generators). For the case in question, one sets L to be the Leech lattice, which has rank 24."
2) "The orbifold construction. In physical terms, this describes a bosonic string propagating on a quotient orbifold. The construction of Frenkel–Lepowsky–Meurman was the first time orbifolds appeared in conformal field theory. Attached to the –1 involution of the Leech lattice, there is an involution h of VL, and an irreducible h-twisted VLmodule, which inherits an involution lifting h. To get the Moonshine Module, one takes the fixed point subspace of h in the direct sum of VL and its twisted module."

Generalized Moonshine:
1) Each V(g) is a graded projective representation of the centralizer of g in M.
2) Each f(g,h,τ) is either a constant function, or a Hauptmodul.
3) Each f(g,h,τ) is invariant under simultaneous conjugation of g and h in M, up to a scalar ambiguity.
4) For each (g,h), there is a lift of h to a linear transformation on V(g), such that the expansion of f(g,h,) is given by the graded trace.
5) $f(g, h, \tau)$ is proportional to J if and only if g = h = 1.

3.3.3 Properties

Remark: The Monster Group can be thought of as a Galois Group over the rational numbers.
Remark: The Monster Group has order: $2^46 \times 3^{20} \times 5^9 \times 7^6 \times 11^2 \times 13^3 \times 17 \times 19 \times 23 \times 29 \times 31 \times 41 \times 47 \times 59 \times 71$ Remark: The Monster Group has 194

conjugacy classes.

3.4 Pariah Group

3.4.1 Background

The Pariah Group was introduced by Robert Griess in Griess (1982) to refer to the six sporadic simple groups which are not subquotients of the monster group. He also named the remaining 26 groups as Happy Family, as these are all subquotients of Monster Group. For example, the orders of J4 and the Lyons Group Ly are divisible by 37. Since 37 does not divide the order of the monster, these cannot be subquotients of it; thus J4 and Ly are pariahs. Four other sporadic groups were also shown to be pariahs by Griess in 1982. The last group J1 was found by Robert A. Wilson in 1986 and this is subgroup of O'N.

4 Bibliography