Representation Theory

Rajeev Sharma

May 2020

Chapter 1

Introduction

Representation Theory is a branch of both Abstract and Linear Algebra that studies algebraic structures. Specifically, this field is concerned with representing these structures as linear transformations of vector mappings. In doing so, an algebraic structure can become more concrete, thus allowing for easier and more concise analysis. This is first achieved by describing elements as matrices. Algebraic operators can also be used to describe binary operations, with the key example being that of matrix addition and multiplication.

There are three main algebraic structures that can be transformed into linear matrices. They are: groups, associative algebraic structures, and Lie algebras. The most studied part of this sub-field is representation group theory, which results in groups transformation via invertible matrices. The resultant binary operation is that of matrix multiplication.

This method is particularly useful due to its ability to reduce complex problems in abstract algebra to simpler problems in linear algebra, thus allowing for the application of simpler, well understood theorems. The following are the three subfields to be detailed within this paper.

Chapter 2

Representation Theory: an Overview

2.1 Representation Theory of Groups

A representation of a group G on a vector space V, mapped over a field K is a group homomorphism F: G $\rightarrow$ GL (V). This is within the general linear group over V, representing all the group automorphisms. This represents the map ρ: G $\rightarrow$ GL(V).

Within this, ρ $(g_1 g_2) = \rho$ (g_1) ρ (g_2) $\forall$g$_1 g_2$ ϵ G.

In this specific example, V is referred to as the representation space. The dimension of V within this specific field is referred to as a representation dimension.

The kernel is not trivial here. Instead, it is defined as ker rho = g ϵ | rho(G) = identity.

2.2 Representation Theory of Associative Algebraic Structures

This specific structure involves a new algebraic structure known as a module. In specific terms, this is a generalization of a vector space over a field. In algebraic terms, this involves rings, which can be described as fields lacking commutativity within multiplication and multiplicative inverses.

Chapter 3

Maschke's Theorem

A particularly important part of group representaiton theory is Maschke's Theorem. First researched by the German mathematician Heinrich Maschke, the theorem builds off of advances from papers published by the Prussian mathematician Felix Klein. A complete proof and analysis of the theorem follows.

3.1 Introductory Definitions

Let G be a group, and let F be $\mathbb{R}$ or $\mathbb{C}$. Also note that GL(n,F) is a group containing invertible $n \times n$ matrices. All results are in F.

A representation of G over F is a homomorphism ρ from G to GL(n,F), for some specific n. We define a degree of ρ as the integer n.

Let's let B be a dihedral group, specifically the D_8 group. Here, this can be defined as (a, b : a^4) = ($b^2 = 1, b^{-1}ab = a^{-1}$).

Our matrices can be defined as such.

A = $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

B = $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Then, note the homomorphism:

ρ: a$^i bj \to A^i B^j (0 \leqslant i \leqslant 3, 0 \leqslant j \leqslant 1)$.

This can be noted as a representation of our group over F. Our degree of ρ is 2.

Let V be a vector space over F, and also let G be a group. We state that V is an FG module if we can specifically define a multiplicative property such that vg has v and g, with both being elements of groups V and G respectively. Furthermore, a series of conditions must be followed.

1. vg ϵ v
2. v(gh) = (vg)h
3. v*1 = v
4. $(\lambda v)g = \lambda(vg)$
5. (u + v)g = ug + vg

.

Through this, we know that the function v is an endomorphism of V.

Let G be a finite group, and let F be $\mathbb{R}$ or $\mathbb{C}$. Let V be an FG-module. If we let U be an FG-submodule of V, then there is an FG-submodule (W) of our group V such that V = U θ W.

Begin by choosing a space within V, titled W_0, s.t V = U θ W_O. Note that $W_0 = sp(v_{m+1}, ..., v_n)$.

For all v ϵ V, there are vectors u ϵ U and w ϵ W, such that u + w = v. We can also define ϕ V $\rightarrow$ V, which can be achieved by writing vϕ= u. We can also write that *pi* is an endomorphism of V by some basic algebra which follows below.

pi:V $\rightarrow$ V, such that (u+w)π = u, $\forall$ uϵ U, w ϵ W.

The image of π is U, and the kernel of π is W.

Now, let's define τ: V $\rightarrow$ V.

vτ = $1/|G|$

$$\sum_{gEG} vg = 1$$

Thus, τ is an endomorphism of V.

Thus, we solely need to prove the existence of τ as an FG homomorphism.

Write that τ is a projection with an image named U. To show this, we can write that τ^2 is τ. Given u ϵ U, and g ϵ G, we can get ug ϵ U, thus making (ug)ϕ=ug. With some algebra, uτ can be shown to equal u.

Letting v ϵ V, and vτ ϵ U, we get (vτ)τ = vτ. Thus, we complete our proof and show that $\tau^2 = \tau$.