MATRIX LIE GROUPS

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Abstract. This paper will focus on the simpler aspects of matrix Lie groups — including the matrix exponential and Lie algebras, finishing with Ado's theorem to prove Lie's Third Theorem.

Lie groups connect almost all branches of mathematics, including representation theory, harmonic analysis, algebraic topology, algebraic geometry, combinatorics, differential geometry, differential equations, number theory, low-dimensional topology, Riemannian geometry, invariant theory, and finite group theory.

Lie groups now play a major role in other non-math fields; representation theory is used extensively in particle physics; Lie groups and Lie algebras are often used in computer vision and finance.

Lie's original motivation for introducing Lie groups was to model the continuous symmetries of differential equations.

1. Preliminaries

Here, we define several basic concepts and go through a few examples that will be used throughout this paper.

Definition 1.1. Denote the space of all $n \times n$ matrices with entries in ring R by $M_n(R)$.

There is not much interesting about $M_n(R)$. It is a ring under matrix addition and multiplication, but it has zero divisors. Non-invertible matrices $A \in R^{n\times n}$ are zero divisors. This can be observed by taking a vector $\vec{v} \in R^n$ with $A\vec{v} = \mathbf{0}$ and filling up a matrix B with $B_{ij} = v_i$. Then $AB = 0$.

Thus, we focus on the multiplicative group of $M_n(R)$:

Definition 1.2. The *general linear group* of degree n over a ring R, denoted $GL(n, R)$, is the set of all $n \times n$ invertible matrices with entries in V.

Let $A, B \in GL(n, R)$, so $\det(A), \det(B) \neq 0$. Hence $\det(AB) = \det(A)\det(B) \neq 0$, hence $AB \in GL(n, R)$. Furthermore, we have

$$1 = \det(I) = \det(A)\det(A^{-1}) \implies \det(A^{-1}) \neq 0 \implies A^{-1} \in GL(n, R),$$

so GL is indeed a group.

We may equip $M_n(R)$ with a norm $\|\cdot\|$, defined as

$$A \mapsto \sqrt{\sum a_{ij}^2}. \tag{1}$$

Date: June 4, 2020.

Thus we can define a distance between two matrices A, B by $d(A,B) = \|A - B\|$. In light of this discussion:

Definition 1.3. A sequence of matrices A_n *converges* to A if $\|A_n - A\| \to 0$ as $n \to \infty$.

Note that since the distance between a_{ij} and b_{ij} is included in this norm, in order for $\|A_n - A\|$ to converge to 0, A_n must also converge entry-wise to A.

Definition 1.4. A *matrix Lie group* is a closed subgroup G of $GL(n, R)$. In other words, if A_n is a sequence of matrices in G converging to a matrix A, then either $A \in G$ or A is not invertible.

It is obvious that the general linear group is a matrix Lie group. Below, we will investigate a few well-known matrix Lie groups.

Definition 1.5. The *special linear group* $SL(n, R)$ consists of all $n \times n$ matrices with entries in R with determinant 1.

This is fairly obviously a subgroup of GL; it is closed under multiplication and inversion by the same calculations following Definition 1.2. It is also a matrix Lie group by continuity of the determinant; $\det(A) = 1 \implies A \in SL(n, R)$.

Definition 1.6. Let $A \in GL_n(\mathbb{C})$, and define $A^* = (\overline{a_{ji}})$ to be the *adjoint* of A. The *unitary group* $U(n) = \{A : A^*A = I\}$ is a subgroup of $GL(n, \mathbb{C})$. We may also define the subgroup $SU(n) \leq U(n)$ consisting of A with determinant 1, called the *special unitary group*.

First we show that $U(n)$ is indeed a subgroup of $GL(n, \mathbb{C})$. Let $A, B \in U(n)$. Then

$$(AB)^*(AB) = B^*A^*AB = B^*B = I \implies AB \in U(n)$$
$$(A^{-1})^*(A^{-1}) = (A^*)^*(A^*) = AA^* = AA^{-1} = I \implies A^{-1} \in U(n)$$

Also notice that since

$$1 = \det(I) = \det(A^*A) = \det\left(\overline{A^\intercal}\right)\det(A) = \overline{\det(A^\intercal)}\det(A) = \overline{\det(A)}\det(A) = |\det(A)|^2,$$

we must have $|\det(A)| = 1$. Consider A_n, a sequence in $U(n)$. Their determinants all Lie on the unit circle; so does the limit of their determinants. Hence $A \in U(n)$, and the unitary group is a matrix Lie group. The same can be said of $SU(n)$.

Definition 1.7. Define the *orthogonal group*

$$O(n, \mathbb{R}) = \left\{A : A \in \mathbb{R}^{n \times n}, A^\intercal A = I\right\}$$

The *special orthogonal group* $SO(n, \mathbb{R}) \leq O(n, \mathbb{R})$ contains only the matrices of positive determinant.

Now we can check that this is a group. Let A, B be orthogonal. Then

$$(AB)^\intercal AB = B^\intercal A^\intercal AB = B^\intercal B = I \implies AB \in O(n, \mathbb{R})$$
$$(A^{-1})^\intercal A^{-1} = (A^\intercal)^{-1}A^{-1} = (AA^\intercal)^{-1} = I^{-1} = I \implies A^{-1} \in O(n, \mathbb{R})$$

To show that it is a matrix Lie group, observe

$$1 = \det(I) = \det(A^\intercal A) = \det(A^\intercal)\det(A) = \det(A)^2,$$

so $\det(A) = \pm 1$. If $A_n \to A$, then $\det(A_n) \to \det(A)$, so $\det(A_n)$ is eventually constant.

Notice that SO is in fact a subgroup of O, as it is closed under multiplication and inversion and is a matrix Lie group for the same reasons as above.

2. The Matrix Exponential and Logarithm

The exponential function for matrices is instrumental in the study of Lie groups. It is used in the definition of a Lie algebra from a Lie group and gives information about the Lie group from its algebra. It is an analog of the exponential function for complex numbers:

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \tag{2}$$

We now prove a few properties of the norm defined in (1) that will allow us to further characterize exp.

Proposition 2.1. *The following are properties of* $\|\cdot\|$*:*

(1) $\|A + B\| \leq \|A\| + \|B\|$
(2) $\|AB\| \leq \|A\| \, \|B\|$
(3) exp *converges under the norm*
(4) $\exp(A)$ *is a continuous function in* A.

Proof. (1) follows from the Triangle Inequality in $\mathbb{C}^{n^2}$, and (2) from the Cauchy-Schwarz Inequality. (3) is true because

$$\sum_{k=0}^{\infty} \left\| \frac{A^k}{k!} \right\| \leq \sum_{k=0}^{\infty} \frac{\|A\|^k}{k!}$$

converges. For (4), let R be a positive real number and $B(R) = \{A \in \mathbb{R}^{n \times n} : |A| \leq R\}$. Let $M_n = R/n!$. Then

$$\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} \frac{R}{n!} = e^R$$

and

$$\left\| \frac{A^n}{n!} \right\| = \frac{\|A^n\|}{n!} \leq \frac{\|A\|^n}{n!} = \frac{R^n}{n!}$$

for each $A \in B(R)$. Then $\exp(A)$ is continuous on $B(R)$ by the Weierstraß M-test. For each $X \in \mathbb{R}^{n \times n}$, there exists an R such that $X \in B(R)$, hence exp is continuous on $\mathbb{R}^{n \times n}$. ■

We also enumerate a few important properties of the exponential function:

Proposition 2.2. *Let* $A, B \in \mathbb{C}^{n \times n}$*. Then*

(a) $e^0 = I$
(b) $(e^A)^* = e^{A*}$
(c) If $AB = BA$*, then* $e^{A+B} = e^A e^B$
(d) e^A *is invertible and* $\left(e^A\right)^{-1} = e^{-A}$
(e) If B *is invertible, then* $e^{BAB^{-1}} = Be^A B^{-1}$.

Proof. (1) is quite obvious, noting that 0^0 is an empty product and thus equal to I. (2) follows by taking term-wise adjoints, which can be done as $(A+B)^* = (\overline{(a+b)_{ji}}) = (\overline{a_{ji}}) + (\overline{b_{ji}}) = A^* + B^*$. (4) is true because

$$e^A e^B = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{A^i}{i!}\cdot\frac{B^j}{j!} = \sum_{s=0}^{\infty}\sum_{i=0}^{s} \frac{A^i}{i!}\cdot\frac{B^{s-i}}{(s-i)!} = \sum_{s=0}^{\infty}\frac{1}{s!}\sum_{i=0}^{s}\binom{s}{i}A^i B^{s-i} \tag{3}$$

It follows from commutativity of A, B that the inner sum equals $(A+B)^s$. Thus we finish (3):

$$e^A e^B = \sum_{s=0}^{\infty}\frac{(A+B)^s}{s!} = e^{A+B}.$$

For (5), we have

$$(BAB^{-1})^n = BAB^{-1}BAB^{-1}\cdots BAB^{-1} = BAA\cdots AB^{-1} = BA^nB^{-1},$$

from which the result follows directly:

$$e^{BAB^{-1}} = \sum_{n=0}^{\infty}\frac{(BAB^{-1})^n}{n!} = \sum_{n=0}^{\infty}\frac{BA^nB^{-1}}{n!} = B\left(\sum_{n=0}^{\infty}\frac{A^n}{n!}\right)B^{-1} = Be^AB^{-1}. \qquad \blacksquare$$

The exponential function has a local inverse around the identity matrix I as well, defined as

$$\log A = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(A-I)^n \tag{4}$$

Note that if $\|A-I\| < 1$, then (4) will converge because $\|(A-I)^n\| \le \|A-I\|^n$ for $n \ge 1$.

Remark. It is also possible for this to converge when $\|A-I\| > 1$, but this doesn't matter too much.

It is also worth noting a few important properties of the logarithm. We state without proof that $e^{\log A} = A$ for $\|A-I\| < 1$ and that $\log(e^B) = B$ for $\|B\| < \log 2$. It immediately follows from the former that $\log(A) + \log(B) = \log(AB)$.

3. Tangent Spaces

The importance of the exponential function is that it maps $T(G)$ to G for a matrix group G. A simple "almost-example" of this is the unit circle in $\mathbb{C}$. We will see that tangent space at the identity (1) is $1 + i\theta$; the exponential function is $e \cdot e^{i\theta}$ is a circle centered at 0 with radius e, which has the same "geometry" as the unit circle.

First, we must define a path:

Definition 3.1. A *path* in G is a map $\gamma : \mathbb{R} \to G$ that sends a variable t to a matrix $A(t) = (a_{ij})$.

In a *smooth* path, each entry, is differentiable in t, in which case we write $A'(t) = (a'_{ij})$. It is useful to consider the set of all matrices $A'(0)$ where $A(0) = I$, called the *tangent space at the identity*.

Proposition 3.2. *$T(G)$ is a vector space.*

This proof is due to [2].

Proof. Let $A'(0) \in T(G)$ and $c \in \mathbb{R}$. Note that $A(ct)$ is also a path through G, and its derivative at 0 is $cA'(ct)$. Since it passes through the identity ($A(c \cdot 0) = A(0) = I$), it follows that $\frac{d}{dt}A'(ct)\big|_{t=0} = cA'(0) \in T(G)$.

Now suppose that $A'(0), B'(0) \in T(G)$; $A(0) = B(0) = I$. Then $A(t)B(t)$ is equal to $I^2 = I$ at 0. Its derivative is $A'(t)B(t) + A(t)B'(t)$, which at 0 is $A'(0)I + IB'(0) = A'(0) + B'(0) \in T(G)$. ∎

We may also define *bracket notation* as $[A, B] = AB - BA$, which we will see gives rise to the Lie algebra. Its foremost property is as follows:

Proposition 3.3. *Let $A, B \in T(G)$. Then $[A, B] \in T(G)$.*

This proof is due to [2].

Proof. Define $\gamma(s, t) = e^{sA}e^{tB}(e^{sA})^{-1} = Ae^{tB}A^{-1}$, a smooth path with $\gamma(0) = 1$. Hence $\frac{d}{dt}\gamma_{t=0} = A(Be^0)A^{-1} \in T(G)$. Also note that $\eta(s) = \frac{d}{dt}\gamma|_{t=0} = e^{sA}e^{tB}(e^{sA})^{-1} = e^{sA}Y(e^{sA})^{-1}$ is a smooth function of s, so its tangent is in $T(G)$. Since

$$\frac{d}{ds}\eta(s) = \frac{d}{ds}(e^{sA}Ye^{-sA}) = (Ae^{sA})(Be^{-sA}) + (e^{sA}B)(-e^{sA}),$$

$\eta'(0) = XY - YX \in T(G)$. ∎

Proposition 3.4. *The Lie bracket also satisfies:*

(1) *Alternativity:* $[A, A] = 0$
(2) *Anti-commutativity:* $[A, B] = -[B, A]$
(3) *Linearity:* $[A + B, C] = [A, C] + [B, C]$
(4) *The Jacobi Identity:* $\sum_{\text{cyc}}[A, [B, C]] = 0$
(5) *Flexibility:* $[A, [B, A]] = [[A, B], A]$.[1]

Proof. (1) is obvious as $[A, A] = AA - AA = 0$. For (2), $[A, B] + [B, A] = AB - BA + BA - AB = 0$. For (3), $[A + B, C] = (A + B)C - C(A + B) = AC + BC - CA - CB = AC - CA + BC - CB = [A, C] + [B, C]$. For (4),

$$\begin{aligned}\sum_{\text{cyc}}[A, [B, C]] &= \sum_{\text{cyc}} A[B, C] - [B, C]A \\ &= \sum_{\text{cyc}} A(BC - CB) - (BC - CB)A \\ &= \sum_{\text{cyc}} ABC - ACB - BCA + CBA.\end{aligned}$$

[1]Note that the bracket is not necessarily associative. Nonetheless, much of the terminology used in the study of Lie algebras also appLies to associative rings and algebras.

It is not hard to check from expanding the sum that this is equal to 0. (5) is again just expansion,

$$\begin{aligned}[A,[B,A]] &= [A, BA - AB] - [AB - BA, A] = ABA - AAB - BAA - ABA \\ &= ABA - BAA - AAB + ABA = [AB - BA, A] = [[A,B],A]. \end{aligned}$$

■

4. Lie Algebras

We now apply the bracket notation in the previous section to develop the idea of Lie algebras and their connection to Lie groups.

Definition 4.1. The *Lie algebra* defined on $T(G)$ is the tangent space equipped with linear operations and the bracket operation. It is customary to denote it with a lowercase fraktur letter (e.g. $\mathfrak{g}$).

Note that given the Lie algebra, one can recover the Lie group as the image of the exponential map (2). That is, $G = \exp(\mathfrak{g})$.

Definition 4.2. A *Lie subalgebra* is a subspace $\mathfrak{h} \subseteq \mathfrak{g}$ which is closed under the Lie bracket.

Definition 4.3. An *ideal* $\mathfrak{i} \subseteq \mathfrak{g}$ is a subalgebra satisfying $[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}$.

Ideals satisfy a stronger condition:

Example. Consider the Lie algebra of diagonal 2×2 matrices $\mathfrak{d}(2) \subset \mathfrak{gl}(2)$. Their bracket is

$$\begin{aligned}\left[\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right] &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} - \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} ax & by \\ cx & dy \end{bmatrix} - \begin{bmatrix} ax & bx \\ cy & dy \end{bmatrix} \\ &= \begin{bmatrix} 0 & b(y-x) \\ c(x-y) & 0 \end{bmatrix},\end{aligned}$$

which is of course not diagonal. However, it is easy to see that the bracket of two diagonal matrices is also diagonal, as in this case we would have $b = c = 0$. Thus $\mathfrak{d}(2)$ is a subalgebra, but not an ideal.

Proposition 4.4. *The tangent space $T(H)$ of a normal subgroup H of a matrix group G is an ideal in $T(G)$.*

Proof. $T(H)$ is a vector space and a subspace of $T(G)$ because any tangent to H at 0 is also a tangent to G at 0. Thus it remains to show that $T(H)$ is closed under Lie brackets. Let $X \in T(G), Y \in T(H)$. Since H is normal, it is closed under conjugation, so we may define an η as in the proof of Proposition 3.3 with tangent vector $XY - YX$. ■

Now, we saw in Proposition 2.2 that $e^{A+B} = e^A e^B$ where A and B commute. Suppose that we instead try to solve $e^A e^B = e^C$ for C. Then $C = \log(e^A e^B)$; expansion shows that

$$\begin{aligned}C &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left[\left(\sum_{i=0}^{\infty} \frac{A^i}{i!} \right) \left(\sum_{j=0}^{\infty} \frac{B^j}{j!} \right) - 1 \right]^k \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\sum_{i+j\geq 1} \frac{A^i B^j}{i!j!} \right)^k \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{i_\ell + j_\ell \geq 1\, \forall \ell \in [k]} \frac{A^{i_1} B^{j_1} \cdots A^{i_\ell} B^{j_\ell}}{i_1! j_1! \cdots i_\ell! j_\ell!}.\end{aligned}$$

We may also form similar developments by taking the derivative of the exponential function, manipulating it, and integrating; this result is known as Dynkin's Formula.[2] We omit the details of its proof.

Proposition 4.5 (Dynkin)**.**

$$C = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{\ell + j_\ell \geq 1\, \forall \ell \in [k]} \frac{[A^{i_1} B^{j_1} \cdots A^{i_k} B^{j_k}]}{(i_1 + j_1 + \cdots + i_k + j_k)(i_1! j_1! \cdots i_k! j_k!)},$$

where we have used the shorthand notation

$$A^{i_1} B^{j_1} \cdots A^{i_k} B^{j_k} = \underbrace{[A, [A, \cdots [A,}_{i_1} \underbrace{[B, [B, \cdots [B,}_{j_1} \cdots \underbrace{[A, [A, \cdots [A,}_{i_k} \underbrace{[B, [B, \cdots B}_{j_k}]] \cdots]].$$

The first few terms are well-known and are given by

(1) $A + B$
(2) $\frac{1}{2}(AB - BA) = \frac{1}{2}[X, Y]$
(3) $\frac{1}{12}(A^2B + AB^2 - 2ABA + B^2A + BA^2 - 2BAB) = \frac{1}{12}([A, [A, B]] + [B, [B, A]])$
(4) $\frac{1}{24}(A^2B^2 - 2ABAB - B^2A^2 + 2BABA) = -\frac{1}{24}[B, [A, [A, B]]]$.

In fact:

Theorem 4.6 (Baker-Campbell-Hausdorff [BCH)**.** *] The terms in the series (Proposition 4.5) can all be expressed in terms of nested Lie brackets with rational coefficients.*

Remark. When A, B commute, all of the brackets vanish, and we are left with only the $A+B$ term, and Proposition 2.2(c) follows.

We now state, mostly without proof, some of the most fundamental theorems in Lie theory that are the main results of the paper. They are well beyond the scope of this paper but may be found in [1] and [3]. Before proceeding, we must first define connected Lie subgroups.

Definition 4.7. Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. Then $H \subset G$ is a *connected Lie subgroup* of G if:

(1) H is a subgroup of G
(2) The Lie algebra $\mathfrak{h}$ of H is a Lie subalgebra of $\mathfrak{g}$

[2]This, along with the above expression for C, converge for $\|A\| + \|B\| < \log 2, \|C\| < \log 2$.

(3) Every element of H may be written as $e^{A_1} \cdots e^{A_k}$ for $A_i \in \mathfrak{h}$.

The following are applications of the BCH Theorem 4.6. This treatment of Lie's Third Theorem 4.10 is due to [1].

Theorem 4.8. *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Then there exists a unique connected Lie group H of G with Lie algebra $\mathfrak{h}$.*

The rough idea with its proof is to show that the Lie algebra of H is $\mathfrak{h}$ by proving subsets in two directions with a countability argument.

Theorem 4.9 (Ado)**.** *Every finite dimensional Lie algebra is a Lie algebra of $\mathfrak{gl}(n, \mathbb{R})$ for sufficiently large n.*

Theorem 4.10 (Lie's Third Theorem)**.** *For any finite-dimensional Lie algebra $\mathfrak{g}$¡ there is a unique simply connected Lie group G whose Lie algebra is $\mathfrak{g}$.*

Proof. By Ado's theorem, we may identify $\mathfrak{g}$ with a real subalgebra of $\mathfrak{gl}(n, \mathbb{C})$. By Theorem 4.8, there is a connected Lie subgroup of $GL(n, \mathbb{C})$ with Lie algebra $\mathfrak{g}$. ■

Lie's Third Theorem is a part of the larger Lie group-Lie algebra correspondence, which includes the homomorphisms theorem and the subgroups-subalgebras theorem. It has importance in various fields of math, along with the correspondence between $SU(2)$ and $SO(3)$ in physics.

Acknowledgements. I am very grateful to my instructor Simon Rubinstein-Salzedo and mentor Eric Frankel for guidance in writing this paper.

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