Integrating in Elementary Terms

Mark Takken, Roy Long, Tristan Liu

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1 Introduction

There are many functions for which an antiderivative is important and applicable. For example, the logarithmic integral function $\mathrm{Li}(x) = \int_2^x \frac{dt}{\log(t)}$ is used in analytic number theory, and notably the prime number theorem. Another example is the Gaussian integral of e^{-x^2}, which comes up as the bell curve in probability. However, it is not possible to express these functions in terms of elementary functions, which loosely means functions that are expressible by a composition of algebraic, exponential and logarithmic operations.

The approach taken to classify the complex functions with an elementary antiderivative will be to consider differential fields, which are fields that generalize the notion of the derivative, that is, the addition and product rule for derivatives of real functions. Then, finding antiderivatives will equate to finding solutions in elementary field extensions. We define elementary field extensions, show that differential fields of characteristic 0 can be uniquely extended in an elementary extension field and show that elementary extensions are closed under differentiation. This gives the necessary framework to work with differential field extensions.
We then state and prove Liouville's Theorem, which is one of the key steps in classifying the integrable functions. It gives conditions on what elements α in a field $\mathbb{F}$ can yield solutions for the differential equation $y' = \alpha$ in an elementary extension field of $\mathbb{F}$. The centerpiece of the argument is a corollary to Liouville's Theorem, the Integrability Criterion, which gives a necessary and sufficient condition for a complex function of a single variable to be elementary integrable.

Using the Integrability Criterion, we look at famous examples of functions such as e^{-x^2} and $\frac{1}{\log(x)}$ and show that they are not elementary integrable function by using the previous corollary. Finally, we give an overview of the Risch Algorithm, which provides a method of computing the elementary integral of a function, if it exists.

In order to fully appreciate the following results, the reader should be familiar with single-variable calculus, particularly with basic differentiation and integration techniques. More importantly, the reader should be familiar with basic field theory: understanding what a field is and understanding field extensions. To understand one section of a proof, the reader will need to be familiar with ring homomorphisms and ideals. In order to understand how trigonometric functions can be expressed with exponentials and logarithms, it is helpful to be familiar with

Euler's formula $e^{ix} = i\sin(x) + \cos(x)$. A section of the proof of Liouville's Theorem assumes the Fundamental Theorem of Symmetric Polynomials, which states that any symmetric polynomial in variables $x_1, \ldots, x_n$ is expressible solely in terms of elementary symmetric $p_1, \ldots, p_n$, where $p_i = \sum_{j_1 < j_2 < \cdots < j_i} \tau_{j_1}\tau_{j_2}\ldots\tau_{j_i}$. Complex analysis is not necessary, though an understanding of it may help rigorize some aspects of the argument. We mostly rely on algebraic techniques to prove our results.

It is our goal in this paper to completely prove each relevant result. However, the casual reader may skip some of the proofs and still appreciate the general argument.

2 Differential Fields

Definition 2.1. Let $\mathbb{F}$ be a field. A map $D : \mathbb{F} \to \mathbb{F}$ is a derivation on $\mathbb{F}$ if, for all $a, b \in \mathbb{F}$, $D(a + b) = D(a) + D(b)$ and $D(ab) = aD(b) + D(a)b$. We also denote $D(a)$ as a'. We call a field $\mathbb{F}$ with a derivation $D_{\mathbb{F}}$ a differential field. A differential subfield is a subfield closed under differentiation. The set of constants of $\mathbb{F}$ are the elements which have derivative 0. We use the notation $D(x)$ and x' interchangeably.

Remark 2.2. We are inspired by ordinary calculus to create a generalization of the derivative, our rule $D(ab) = aD(b) + D(a)b$, Leibniz's rule, being analogous to the product rule. Creating this algebraic definition of the derivative will allow us to determine a precise (algebraic) criterion for integrability in elementary terms.

Proposition 2.3. *For all differential fields $\mathbb{F}$, we have:*

1. $D(1) = 0$
2. *Power rule:* $D(x^n) = nx^{n-1}D(x)$ *for all* $n \in \mathbb{Z}, x \in \mathbb{F}$
3. *Quotient rule:* $D(\frac{x}{y}) = \frac{yD(x) - xD(y)}{y^2}$ *for all* $x, y \in \mathbb{F}$

Proof.

1. $D(1) = D(1 \cdot 1) = 1 \cdot D(1) + D(1) \cdot 1 = D(1) + D(1)$, which implies $D(1) = 0$.
2. We prove this for positive n by induction. The base case $n = 1$ is clearly true, as $D(x^1) = 1 \cdot x^0 \cdot D(x)$. For the inductive step, suppose that $D(x^{n-1}) = (n-1)x^{n-2}D(x)$. Then:

   $$D(x^n) = D(x^{n-1} \cdot x) = x^{n-1} \cdot D(x) + D(x^{n-1}) \cdot x = (x^{n-1} + (n-1)x^{n-1})D(x) = nx^{n-1}D(x),$$

   as desired. We proceed to prove this for negative n as well. Consider an element x^{-n}, with $n \in \mathbb{Z}^+$. We then have:

   $$0 = D(1) = D(x^n \cdot x^{-n}) = x^n D(x^{-n}) + D(x^n)x^{-n} = x^n D(x^{-n}) + D(x^n)x^{-n}$$

   Solving for $D(x^{-n})$ gives us:

   $$D(x^{-n}) = -D(x^n)x^{-2n} = -nx^{n-1}D(x)x^{-2n} = -nx^{-n-1}D(x),$$

   as desired.

3. We treat $\frac{x}{y}$ as the product $x \cdot y^{-1}$ and simplify:

$$D\Big(\frac{x}{y}\Big) = D(x \cdot y^{-1}) = xD(y^{-1}) + y^{-1}D(x) = -xy^{-2}D(y) + y^{-1}D(x) = \frac{yD(x) - xD(y)}{y^2}.$$

□

Thus we have that the rules of the derivation are analogous to the derivative rules in calculus.

Proposition 2.4. *The set of constants is a differential subfield of* $\mathbb{F}$*.*

Proof. It is easy to verify that the field of constants is closed under addition, multiplication, inversion and differentiation, so it is a differential subfield. □

Example. $D(a) = 0$ for all $a \in \mathbb{F}$ is always a derivation, but it is not a very interesting one. The set of constants is just all of $\mathbb{F}$.

Example. It follows from the fact that $D(1) = 0$ and from the additive property of the derivation that if the integers are contained in a certain field, then the derivation of any integer is 0. It further follows from the quotient rule that if the rationals are contained in a certain field, then the derivation of any rational number is 0.

Example. $\mathbb{C}(x)$ under the d/dx derivation is a differential field. The constants are $\mathbb{C}$.

Example. $\mathbb{Q}(e)$ under a "d/de" derivation is a differential field. (By this we mean the derivative of e would be 1 and the derivative of e^{17} would be $17e^{16}$, etc). The constants are $\mathbb{Q}$.

3 Differential and Elementary Field Extensions

Definition 3.1. A differential extension field of a differential field $\mathbb{F}$ is an extension field which extends the derivation on $\mathbb{F}$.

$\mathbb{F}(t)/\mathbb{F}$ is a logarithmic extension if there exists $x \in \mathbb{F}$ with $t' = \frac{x'}{x}$. We say $\mathbb{F}(t)/\mathbb{F}$ is an exponential extension if there exists some $x \in \mathbb{F}$ with $t' = tx'$.

Remark 3.2. Our definition is analogous to exponentials and logarithms in calculus because they satisfy the same differential equations.

Definition 3.3. A differential extension field $\mathbb{E}$ of a differential field $\mathbb{F}$ is elementary if it can be formed by a sequence of algebraic, logarithmic, or exponential extensions. In other words, there exists a tower of differential fields

$$\mathbb{F} = \mathbb{F}_0 \subseteq \mathbb{F}_1 \subseteq \mathbb{F}_2 \subseteq \cdots \subseteq \mathbb{E}$$

such that each $\mathbb{F}_i/\mathbb{F}_{i-1}$ is an algebraic, logarithmic or exponential extension. A function is said to be elementary if it is in an elementary extension of $\mathbb{C}(x)$.

Example. Trigonometric functions are elementary. Note that $\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$ follows Euler's Formula and can be inverted to yield $\arcsin(x) = -i\log(ix + \sqrt{1 - x^2})$. Similar formulas can be found for $\cos(x), \tan(x), \arccos(x)$, and $\arctan(x)$.

Remark 3.4. Note that some elementary functions are not familiar to us. For example, $f(c)$ defined to be the unique real root of the polynomial $x^{17}+17x+c$ is an elementary function.

We will use the following theorem when proving Liouville's Theorem in the next section.

Theorem 3.5. *For a differential field $\mathbb{F}$ of characteristic 0 and an algebraic extension field $\mathbb{K}/\mathbb{F}$, a derivation on $\mathbb{F}$ can be uniquely extended to $\mathbb{K}$.*

Proof. Let $\alpha \in \mathbb{K}$, and let $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in \mathbb{F}[X]$ be the minimal polynomial for α over $\mathbb{F}$. Define $M_0, M_1 : \mathbb{F}[X] \to \mathbb{F}[X]$ by:

$$M_0\Big(\sum_{i=0}^{m} b_iX^i\Big) = \sum_{i=0}^{m} b_i'X^i,$$
$$M_1\Big(\sum_{i=0}^{m} b_iX^i\Big) = \sum_{i=1}^{m} ib_iX^{i-1}.$$

Note that M_1 is the formal derivative, as in ordinary calculus. Further note that $(M_1f)(X)$ is nonzero because $\mathbb{F}$ has characteristic 0 and has lesser degree than f, so $(M_1f)(\alpha) \neq 0$. If there does exist an extension of the derivative to $\mathbb{K}$, we then have

$$\begin{aligned} f(\alpha)' &= n\alpha^{n-1}\alpha' + (a_{n-1}'\alpha^{n-1} + (n-1)a_{n-1}\alpha^{n-2}\alpha') + \cdots + (a_1'\alpha + a_1\alpha') + a_0' \\ &= (M_1f)(\alpha)\alpha' + (M_0f)(\alpha). \end{aligned}$$

We thus have $0 = 0' = f(\alpha)' = (M_1f)(\alpha)\alpha' + (M_0f)(\alpha)$, so $\alpha' = -\frac{(M_0f)(\alpha)}{(M_1f)(\alpha)}$. Therefore, the uniqueness of an extension of the derivation to $\mathbb{K}$ follows from its existence.

We now prove existence. Let $\mathbb{K} = \mathbb{F}(\alpha)$, for a particular $\alpha \in \mathbb{K}$. For some $g(X) \in \mathbb{F}[X]$, to be determined later, define the map $M : \mathbb{F}[X] \to \mathbb{F}[X]$ to be given by

$$MA = M_1A \cdot g(X) + M_0A,$$

for any $A \in \mathbb{F}[X]$. (Note that this definition can be motivated by the above expression for $f(\alpha)'$; $g(X)$ is analogous to α'.) We have that $M(A+B) = MA + MB$ and $M(AB) = (MA)B + A(MB)$, since, as it can be easily verified, these identities hold for M_0 and M_1. In addition, we have $Db = b'$ for all $b \in \mathbb{F}$.

Consider the surjective ring homomorphism $R : \mathbb{F}[X] \to \mathbb{F}[\alpha]$, which sends any element of $\mathbb{F}$ to itself and sends X to α. Since $\mathbb{F}[\alpha] = \mathbb{F}(\alpha) = \mathbb{K}$ (because α is algebraic), M will induce a derivation on $\mathbb{K}$ extending that on $\mathbb{F}$ if M maps the kernel of R to itself. But this kernel is the ideal $\mathbb{F}[X]f(X)$, that is, all multiples of $f(X)$, where $f(X)$ is the minimal polynomial of α over $\mathbb{F}$. Now, the condition that M map $\mathbb{F}[X]f(X)$ onto itself is equivalent to that M map $f(X)$ to a multiple to itself, which in turn is equivalent to that Mf be an element of $\mathbb{F}[X]$ of which α is a root, which finally is equivalent to $Mf(\alpha) = M_1f(\alpha)\cdot g(\alpha) + M_0f(\alpha) = 0$. Because $M_1f(\alpha) \neq 0$ and $\mathbb{F}(\alpha) = \mathbb{F}[\alpha]$, we can solve for $g(\alpha)$. This gives us a desired $g(X) \in \mathbb{F}[X]$ that maps the kernel of R to itself. Therefore, there exists an extension of the derivative from $\mathbb{F}$ to $\mathbb{K}$, and this proves the theorem.

□

Example. Let $\mathbb{K} = \mathbb{Q}(\sqrt{17})$. We show how to compute $D(\sqrt{17})$. The minimal polynomial of $\sqrt{17}$ is $x^2 - 17$. We then have that $0 = (\sqrt{17}^2 - 17)' = 2\sqrt{17}D(\sqrt{17})$, and thus $D(\sqrt{17}) = 0$.

Example. Let $\mathbb{F} = \mathbb{C}(x)$ and $\mathbb{K} = \mathbb{F}(\sqrt{x})$. We show how to compute $D(\sqrt{x})$ given x'. The minimal polynomial of $\sqrt{x}$ is $f(y) = y^2 - x$. We then have that $0 = (\sqrt{x}^2 - x)' = 2\sqrt{x}D(\sqrt{x}) - x'$, so $D(\sqrt{x}) = \frac{x'}{2\sqrt{x}}$.

Corollary 3.6. *An elementary extension field of $\mathbb{C}(x)$ is closed under ordinary differentiation given by $x' = 1$. In other words, the derivative is indeed a valid derivation on any elementary extension field of $\mathbb{C}(x)$.*

Proof. Let $K = \mathbb{C}(x, f_1, f_2, ..., f_n)$ be an elementary extension field of $\mathbb{C}(x)$. We prove the corollary by induction on n. For the base case $n = 0$, the fact that $\mathbb{C}(x)$ is closed under differentiation follows from the additive and multiplicative properties of the derivative and the quotient rule. For the inductive step, suppose the corollary is true up to some $n - 1$, that is, $\mathbb{C}(x, f_1, ..., f_{n-1})$ is closed under differentiation. Consider an additional elementary extension $\mathbb{C}(x, f_1, ..., f_{n-1}, f_n)$. If f_n is algebraic, then by the previous theorem, there exists a unique way to extend the existing derivation to $\mathbb{C}(x, f_1, ..., f_{n-1}, f_n)$, which is the derivative. If f_n is an exponential, then $f_n' = f_n\beta' \in \mathbb{C}(x, f_1, ..., f_{n-1}, f_n)$ for some $\beta \in \mathbb{C}(x, f_1, ..., f_{n-1})$, and if f_n is a logarithm, then $f_n' = \frac{\beta'}{\beta} \in \mathbb{C}(x, f_1, ..., f_{n-1}, f_n)$ for some $\beta \in \mathbb{C}(x, f_1, ..., f_{n-1})$. Therefore, $\mathbb{C}(x, f_1, ..., f_{n-1}, f_n)$ is closed under differentiation, and this proves the corollary. □

4 Liouville's Theorem

Lemma 4.1. *Let $\mathbb{F}$ be a differential field with differential field extension $\mathbb{F}(t)$. Suppose $\mathbb{F}(t)$ and $\mathbb{F}$ have the same constants and t is transcendental over $\mathbb{F}$.*

1. *Let $t' \in \mathbb{F}$ and $f(t) \in \mathbb{F}[t]$ with $\deg(f(t)) > 0$. Then $f(t)' \in \mathbb{F}[t]$, and $\deg(f(t)) = \deg(f(t)')$ iff the leading coefficient of $f(t)$ is non-constant. Otherwise, if the leading coefficient of $f(t)$ is constant, $\deg(f(t)') = \deg(f(t)) - 1$.*

2. *Let $\frac{t'}{t} \in \mathbb{F}$. Then for all nonzero $a \in F$ and nonzero $n \in \mathbb{Z}$, $(at^n)' = ht^n$ for some nonzero $h \in F$. Moreover, if $f(t) \in \mathbb{F}[t]$ with $\deg(f(t)) > 0$, then $\deg(f(t)) = \deg(f(t)')$. Also, $f(t)' = cf(t)$ for some $c \in \mathbb{F}$ iff $f(t)$ is a monomial.*

Proof.

1. Suppose $t' \in \mathbb{F}$. Let $f(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_0 \in F[t]$, and $a_n \neq 0$. Then

$$f(t)' = (a_n' t^n + na_n t' t^{n-1}) + (a_{n-1}' t^{n-1} + (n-1)a_{n-1}t't^{n-2}) + \cdots$$

   which clearly lies in $\mathbb{F}[t]$ as $t' \in \mathbb{F}$. The leading coefficient of $f(t)'$ is just a_n', so $\deg(f'(t)) = \deg(f(t))$ iff $a_n' \neq 0$, ie. a_n is not constant. Otherwise, $\deg(f'(t)) = n - 1 = \deg(f(t)) - 1$.

2. On the other hand, if $\frac{t'}{t} = b \in \mathbb{F}$, then we have $(at^n)' = a't^n + nat't^{n-1} = (a' + nab)t^n$. This cannot be equal to zero, as t is transcendental over $\mathbb{F}$. Thus, if $\deg(f(t)) > 0$,

then $\deg(f(t)) = \deg(f(t)')$. Lastly, if $f(t)' = cf(t)$, then $(a' + nab)/a = c$, for any $a \in \{a_0, a_1, \cdots, a_n\}$. Choosing two distinct coefficients a_p, a_q gives

$$\frac{a_p' + pa_pb}{a_p} = \frac{a_q' + qa_qb}{a_q},$$

or $(a_p' + pa_pb)a_q = (a_q' + qa_qb)a_p$. Thus, taking the derivative of the function $\frac{a_pt^p}{a_qt^q}$ gives us

$$\left(\frac{a_pt^p}{a_qt^q}\right)' = \frac{(a_p' + pa_pb)a_qt^{p+q} - (a_q' + pa_qb)a_pt^{p+q}}{a_q^2t^{2q}} = 0,$$

which means $\frac{a_pt^p}{a_qt^q}$ is a constant z, which is in $\mathbb{F}$ because $\mathbb{F}$ and $\mathbb{F}(t)$ have the same constants. However, then $a_pt^p - za_qt^q = 0$, contradicting the given fact that t is trascendental.

□

We are now ready to prove the main result of this section, which provides a necessary (and sufficient, though this is not so important) condition for the existence of an antiderivative of an element of a field lying in an elementary extension field.

Theorem 4.2. *(Liouville) Let $\mathbb{F}$ be a differential field of characteristic 0. Let $\alpha \in \mathbb{F}$. If $y' = \alpha$ has a solution y in an elementary extension of $\mathbb{F}$ with the same constants, then there exist constants $c_1, \cdots, c_n$ and elements $u_1, \cdots, u_n, v \in \mathbb{F}$ such that*

$$\alpha = v' + \sum_{i=1}^{n} c_i \frac{u_i'}{u_i}.$$

Proof. The given elementary extension of $\mathbb{F}$ consists of a tower of algebraic, logarithmic or exponential extensions

$$F \subset F(t_1) \subset F(t_1, t_2) \subset \cdots \subset F(t_1, t_2, \ldots, t_N)$$

all with the same constants. We prove the theorem by induction on N. The base case $N = 0$ is obvious (simply let $v = y$ and all $c_i, u_i = 0$), so assume that $N > 0$ and the theorem holds for $N - 1$. Applying the case $N - 1$ to the fields $\mathbb{F}(t_1) \subset \mathbb{F}(t_1, t_2, \ldots, t_N)$ gives us an expression for α in the desired form, but with $u_1, u_2, \ldots, u_n, v \in \mathbb{F}(t_1)$. We wish to find a similar expression, but with $u_1, u_2, \ldots, u_n, v \in \mathbb{F}$. In the following argument, we let $t = t_1$.

We first consider the case where t is algebraic with minimal polynomial $f \in \mathbb{F}[X]$. There are then polynomials $U_1, U_2, \ldots, U_n, V \in \mathbb{F}[X]$ such that $U_1(t) = u_1, U_2(t) = u_2, \ldots, U_n(t) = u_n, V(t) = v$. (This follows from applying the Euclidean Algorithm to polynomials in $\mathbb{F}[X]$.) Let $\tau_1 (= t), \tau_2, \ldots, \tau_k \in \mathbb{K}$ be the distinct conjugates of t over the splitting field of f. By Theorem 3.5, there is a unique extension of the derivation of $\mathbb{F}$ to $\mathbb{K}$. Thus, since $\alpha = V(t) + \sum_{i=1}^{n} \frac{U_i(t)'}{U_i(t)}$ has degree 0 with respect to t and each τ_j has the same minimal polynomial over $\mathbb{F}$ as t, we have

$$\alpha = V(\tau_j)' + \sum_{i=1}^{n} c_i \frac{U_i(\tau_j)'}{U_i(\tau_j)},$$

for each $j = 1, 2, \ldots, k$. It follows that

$$\begin{aligned}
\alpha &= \frac{1}{k}\sum_{j=1}^{k}\left(V(\tau_j)' + \sum_{i=1}^{n} c_i \frac{U_i(\tau_j)'}{U_i(\tau_j)}\right) \\
&= \frac{1}{k}\sum_{j=1}^{k} V(\tau_j)' + \sum_{i=1}^{n}\frac{c_i}{k}\sum_{j=1}^{k}\frac{U_i(\tau_j)'}{U_i(\tau_j)} \\
&= \left(\frac{V(\tau_1)+\cdots+V(\tau_k)}{k}\right)' + \sum_{i=1}^{n}\frac{c_i}{k}\cdot\frac{(U_i(\tau_1)U_i(\tau_2)\ldots U_i(\tau_k))'}{U_i(\tau_1)U_i(\tau_2)\ldots U_i(\tau_k)}.
\end{aligned}$$

Each $U_i(\tau_1)U_i(\tau_2)\ldots U_i(\tau_k)$ as well as $V(\tau_1)+\cdots+V(\tau_k)$ is a symmetric polynomial in $\tau_1, \ldots, \tau_k$. Let P be an arbitrary such polynomial, and let $p_1, \ldots, p_k$ be the elementary symmetric functions in $\tau_1, \ldots, \tau_k$. That is,

$$p_i = \sum_{j_1 < j_2 < \cdots < j_i} \tau_{j_1}\tau_{j_2}\ldots\tau_{j_i}$$

By the Fundamental Theorem of Symmetric Polynomials, P can be expressed as a series of sums and products of elements of $\mathbb{F}$ and these elementary symmetric functions. But by Vieta's formulas, p_i is the $(i+1)^{\text{th}}$ coefficient of f, which is in $\mathbb{F}$. So $P \in \mathbb{F}$. Therefore, $\frac{(U_i(\tau_1)U_i(\tau_2)\ldots U_i(\tau_k))'}{U_i(\tau_1)U_i(\tau_2)\ldots U_i(\tau_k)}, \frac{V(\tau_1)+\cdots+V(\tau_k)}{k} \in \mathbb{F}$, and we have found a desired expression for α.

Now, suppose t is instead transcendental. Then we have

$$\alpha = v(t)' + \sum_{i=1}^{n} c_i \frac{u_i(t)'}{u_i(t)},$$

with $u_1(t), u_2(t), \ldots, v(t) \in \mathbb{F}(t)$. We examine an arbitrary $u_i(t)$. Let $u_i(t) = \delta p_1(t)^{e_1} p_2(t)^{e_2} \ldots p_r(t)^{e_s}$, where $\delta \in \mathbb{F}$, $e_1, e_2, \ldots, e_s \in \mathbb{Z}$ and $p_1(t), p_2(t), \ldots, p_r(t)$ are distinct, monic and irreducible elements of $\mathbb{F}[t]$. Then:

$$\begin{aligned}
\frac{u_i(t)'}{u_i(t)} &= \frac{(\delta p_1(t)^{e_1} p_2(t)^{e_2} \ldots p_r(t)^{e_s})'}{\delta p_1(t)^{e_1} p_2(t)^{e_2} \ldots p_r(t)^{e_s}} \\
&= \frac{\delta'}{\delta} + \sum_{j=1}^{s} \frac{(p_j(t)^{e_j})'}{p_j(t)^{e_j}} \\
&= \frac{\delta'}{\delta} + \sum_{j=1}^{s} e_j \frac{p_j(t)'}{p_j(t)}
\end{aligned}$$

Thus, after decomposing each $\frac{u_i(t)'}{u_i(t)}$ in such a way, we may assume that each $u_i(t)$ is a monic irreducible element of $\mathbb{F}[t]$ or an element of $\mathbb{F}$, and $u_1(t), u_2(t), \ldots, u_n(t)$ are all distinct. Next, consider the partial fraction decomposition of $v(t)$, which expresses $v(t)$ as the sum of an element of $\mathbb{F}[t]$ and various terms of the form $\frac{g(t)}{f(t)^r}$, where $f(t)$ is a monic irreducible element of $\mathbb{F}[t]$, r is a positive integer, and $g(t)$ is a nonzero element of $\mathbb{F}[t]$ with degree less than that of f. We now split into two subcases: t is a logarithm of an element of $\mathbb{F}$, and t is an exponential of an element of $\mathbb{F}$.

We begin with the former subcase, so that $t' = \frac{a'}{a}$, for some $a \in \mathbb{F}$. Let $f(t)$ be an arbitrary monic irreducible element of $\mathbb{F}[t]$. By Part 1 of Lemma 4.1, $f(t)' \in \mathbb{F}[t]$ with degree one less than that of $f(t)$, so that $f(t)$ does not divide $f(t)'$. In fact, $f(t)$ and $f(t)'$ share no common factors, since $f(t)$ is irreducible over $\mathbb{F}$. Thus, if $u_i(t) = f(t)$ for some i, then $\frac{(u_i(t))'}{u_i(t)}$ is already in lowest terms, with denominator $f(t)$. Now, if a term $\frac{g(t)}{f(t)^r}$ were to occur in the partial fraction decomposition of $v(t)$ described above, with $r > 0$ and maximal among terms having $f(t)$ in the denominator, then $v(t)'$ would contain the term $\left(\frac{g(t)}{f(t)^r}\right)' = \frac{-rg(t)f(t)'}{f(t)^{r+1}}$. Because $f(t)$ does not divide $g(t)$ or $f(t)'$ and $f(t)$ is irreducible, we would have that $f(t)$ does not divide this numerator, and a term with denominator $f(t)^{r+1}$ would appear in $v(t)'$, with $r + 1 >= 2$. Since one of the $u_i(t)$'s can have $f(t)$ in the denominator at most once, this would imply that α consists of a term containing $f(t)$, which is impossible. Thus, $f(t)$ does not appear in the denominator of $v(t)$. In addition, in order for α not to contain $f(t)$, $f(t)$ cannot be equal to $u_i(t)$, for any i. Since $f(t)$ was chosen arbitrarily, each $u_i(t) \in \mathbb{F}$ and $v(t) \in \mathbb{F}[t]$. But then $v(t)'$ must be in $\mathbb{F}$, so Part 1 of Lemma 5.1 implies that $v(t) = ct + d$, with c a constant and $d \in \mathbb{F}$. Therefore,

$$\begin{aligned}
\alpha &= \sum_{i=1}^{n} \frac{(u_i)'}{u_i} + v(t)' \\
&= \sum_{i=1}^{n} \frac{(u_i)'}{u_i} + ct' + d' \\
&= \sum_{i=1}^{n} \frac{(u_i)'}{u_i} + c\frac{a'}{a} + d'
\end{aligned}$$

gives us an expression for α in the desired form.

Now, suppose that t is the exponential of an element of $\mathbb{F}$, so that $\frac{t'}{t} = b'$, for some $b \in \mathbb{F}$. By Part 2 of Lemma 4.1, if $f(t)$ is a monic irreducible element of $\mathbb{F}[t]$ other than t itself, then $f(t)' \in \mathbb{F}$ and $f(t)$ does not divide $f(t)'$. The very same reasoning given above shows that $f(t)$ cannot occur in the denominator of $v(t)$, and no $u_i(t)$ is equal to $f(t)$. Thus each $u_i(t)$ is in $\mathbb{F}$, with the possible exception that one of these may be t. Because each $\frac{u_i(t)'}{u_i(t)} \in \mathbb{F}$ (recall that $\frac{t'}{t}$ is in $\mathbb{F}$), we have $v(t)' \in \mathbb{F}$. Part 2 of Lemma 4.1 thus implies that $v(t) \in \mathbb{F}$. If each $u_i(t) \in \mathbb{F}$ without exception, then α is already written in the desired form, and we are done. Otherwise, there is exactly one $u_i(t)$, without loss of generality let it be $u_1(t)$, that is equal to t. Then

$$\begin{aligned}
\alpha &= v' + c_1\frac{t'}{t} + \sum_{i=2}^{n} c_i\frac{(u_i)'}{u_i} \\
&= (c_1 b + v)' + \sum_{i=2}^{n} c_i\frac{(u_i)'}{u_i},
\end{aligned}$$

with $u_2, u_3, \ldots, u_n$, $c_1 b + v$ all in $\mathbb{F}$, which is in the desired form. This completes the proof of the theorem. □

Remark 4.3. In general, the condition that $\mathbb{F}$ and its elementary extension field have the same constants is important. For example, consider $\mathbb{F} = \mathbb{R}(x)$, and $\alpha = \frac{1}{x^2+1}$. We know that $\int \frac{1}{x^2+1} dx$ is in some elementary extension field of $\mathbb{R}(x)$, but it turns out we cannot write $\frac{1}{x^2+1}$ in the form given by Liouville's Theorem.

To prove this, suppose for contradiction that $\frac{1}{x^2+1}$ can be written in the desired form, with $u_i, v \in \mathbb{R}(x)$ and constants $c_i \in \mathbb{R}$, that is, $\frac{1}{x^2+1} = \sum_{i=1}^n c_i \frac{(u_i)'}{u_i} + v'$. By the same reasoning as given in the transcendental case of Liouville's Theorem above, we may assume that each u_i is a monic and irreducible element of $\mathbb{R}(x)$. Next, note that if x^2+1 appears as a factor of the denominator of v (where the numerator and denominator share no common factors), then it appears as a factor of the denominator of v' with multiplicity greater or equal to 2. But this is impossible, since it appears in the denominator of the left-hand side with multiplicity 1. Thus, x^2+1 does not appear in the denominator of v, and hence it also does not appear in that of v'.

Since the right-hand side must contain a factor of x^2+1 in its denominator, one of the u_i's must be equal to x^2+1; without loss of generality, let it be u_1. We subtract the term $c_1 \frac{(u_1)'}{u_1} = \frac{2c_1 x}{x^2+1}$ from both sides, which gives us $\frac{1-2c_1 x}{x^2+1} = \sum_{i=2}^n c_i \frac{(u_i)'}{u_i} + v'$. The right-hand side no longer contains a factor of x^2+1 it its denominator, so neither does the left-hand side. But this implies that x^2+1 divides $1-2c_1 x$, a polynomial with lesser degree, which is a contradiction.

It turns out that the converse of Liouville's Theorem is also true, and this result is straightforward.

Corollary 4.4. *(Integrability Criterion) For $f, g \in \mathbb{C}(x)$ with $f \neq 0$ and g not a constant, $f(x)e^{g(x)}$ can be integrated in elementary terms iff $f(x) = r'(x) + g'(x)r(x)$ for some $r(x) \in \mathbb{C}(x)$.*

Proof. If $f(x) = r'(x) + g'(x)r(x)$ for some rational function $r(x) \in \mathbb{C}(x)$, then we have that $[r(x)e^{g(x)}]' = r'(x)e^{g(x)} + g'(x)r(x)e^{g(x)} = f(x)e^{g(x)}$, so $f(x)e^{g(x)}$ does have an elementary antiderivative.

We now prove the converse. First, in order to be able to apply the lemma, we show that e^g is transcendental over $\mathbb{C}(x)$. Assume for the sake of contradiction that this is not the case, so that e^g has a minimal polynomial f with

$$f(e^g) = e^{ng} + a_1 e^{(n-1)g} + \cdots + a_n,$$

where $a_1, \ldots, a_n \in \mathbb{C}(x)$. Then differentiating gives us:

$$ng' e^{ng} + (a_1' + (n-1)a_1 g' e^{ng} + \cdots + a_n' = 0.$$

Because f is minimal, this second equation is proportional to the first, so that $ng' = \frac{(a_n)'}{a_n}$. If $(a_n)'$ is nonzero, then a_n can be expressed as the power product of linear terms of $\mathbb{C}[z]$. It follows that we can decompose $\frac{(a_n)'}{a_n}$ into a sum of fractions with constant numerators and linear denominators. (We can do this in the same way the we decomposed $\frac{u_i(t)'}{u_i(t)}$ in Liouville's Theorem.) If g is a polynomial, then clearly ng' has no linear denominator. Otherwise, suppose a term $\frac{c}{f(x)^b}$ appears in its partial fraction decomposition, where c is a constant, $f(x)$ is linear, and $b \geq 1$. Then a term $\frac{-nbcf(x)'}{f(x)^{b+1}}$ appears in the partial fraction decomposition of ng', with $b+1 \geq 2$. Thus, regardless, the partial fraction decomposition of ng' has no term with linear denominator, whereas $\frac{(a_n)'}{a_n}$ does contain terms with linear denominator, a contradiction. Therefore, $(a_n)' = 0$, implying that $ng' = 0$, contradicting the assumption that g is not constant. Thus, e^g is indeed transcendental.

Let $t = e^g$ and $\mathbb{F} = \mathbb{C}(x)$. If $\int fe^g$ is elementary, we then have

$$ft = v' + \sum_{i=1}^{n} c_i \frac{u_i'}{u_i}$$

where the c_i's in $\mathbb{C}$ and the u_i's in $\mathbb{F}(t)$. Expand v into partial fractions with respect to $\mathbb{F}[t]$. By the lemma and the same reasoning as in the last subcase in Liouville's Theorem, we have that t is the only possible monic irreducible factor of a denominator in v. We have that $v = \sum b_j t^j$ for integers j and b_j's in $\mathbb{F}$. Furthermore, we can make it so that all the u_i's are in $\mathbb{F}$ or are distinct monic irreducible elements of $\mathbb{F}(t)$. We then have, again by the same exact reasoning as in Liouville's, that if $u_i \notin \mathbb{F}$, then $u_i = t$. In this case, $\frac{u_i'}{u_i} \in \mathbb{F}$, which means that $\sum_i c_i u_i'/u_i \in \mathbb{F}$. For brevity, we will call this sum q in the following computations.

Then we have

$$ft = \Big(\sum_j b_j t^j\Big)' + q = \sum_j (b_j' t^j + j b_j t' t^{j-1}) + q = \sum_j (b_j' t^j + j b_j g' t^j) + q.$$

The left-hand side and the right-hand side are polynomials in t. Equating the coefficient of t gives $f = b_1' + b_1 g'$. Thus, $r(x) = b_1 \in \mathbb{F}$ gives us the solution we wanted.

□

Remark 4.5. Note that this criterion is not merely stating that this differential equation has a solution, bur rather that there is a solution with the special property that it is a rational function. If it is satisfied, then the function $r(x)e^{g(x)}$ is an elementary anti-derivative of f. However, if such a rational function $r(x)$ does not exist that solves the differential equation $r'(x) + g'(x)r(x) = f(x)$ for some particular f and g, then this verifies that the function $f(x)e^{g(x)}$ does not have an elementary anti-derivative. In the following examples, we will use this useful result to show that certain functions, like $\frac{1}{\log(x)}$ and the Gaussian distribution e^{-x^2}, in fact cannot be elementarily integrated.

5 Examples of Nonintegrable functions

First, we prove a lemma which will be used in later examples.

Lemma 5.1. *Let $P(x)$ be a polynomial. If r is a root of P with multiplicity $e \geq 1$, then r has multiplicity $e-1$ for the polynomial P'.*

Proof. Write $P(x) = c\prod_{i=1}^{n}(x - r_i)^{e_i}$, with $r = r_1$. By the product rule, we obtain

$$P'(x) = e_1(x-r_1)^{e_1-1}\prod_{i=2}^{n}(x-r_i)^{e_i} + (x-r_1)^{e_i}\sum_{i=2}^{n} e_i(x-r_i)^{e_i-1}\prod_{\substack{2\leq j\leq n\\ j\neq i}}(x-r_i)^{e_i}$$

The first term has r as a root with multiplicity e_1-1, and the second term has r with multiplicity e_1. Thus, r is indeed a root of the entire expression with multiplicity $e_1 - 1$. □

We have finally come to the main purpose of our paper: proving that certain functions do not have an elementary antiderivative!

Example. $e^{f(x)}$ is not integrable in elementary functions for for polynomials $f(x)$ with degree greater than 2.

Proof. We use the previous corollary. Showing $e^{f(x)}$ is not integrable is thus equivalent to showing there does not exist a rational function $r(x)$ such that $1 = r'(x) + f'(x)r(x)$. Let $r(x) = \frac{p(x)}{q(x)}$ for relatively prime polynomials p, q. We then have

$$1 = \frac{p'(x)q(x) - p(x)q'(x)}{q(x)^2} + \frac{f'(x)p(x)}{q(x)}$$

which, clearing denominators, is equivalent to

$$q(x)^2 = p'(x)q(x) - p(x)q'(x) + f'(x)p(x)q(x)$$

or

$$p(x)q'(x) = q(x)[p'(x) + f'(x)p(x) - q(x)],$$

where we have moved all the multiples of q to the right hand side.

Suppose $q(x)$ has a root r with multiplicity k dividing $q(x)$. Since $p(x)$ and $q(x)$ are relatively prime, we must have $q'(x)$ has the same root r with multiplicity k, which is a contradiction because differentiating a function reduces the multiplicity of the roots by one. Thus $q(x)$ must be a constant, and $r(x)$ a polynomial.

However, $r'(x) + f'(x)r(x)$ is obviously not a constant polynomial for a polynomial $r(x)$ since f'(x) is nonconstant, and thus cannot equal 1. Our integral then cannot be elementary. □

Specifically, this means that e^{-x^2} is not integrable in elementary terms.

Example. $\frac{e^x}{f(x)}$ is not integrable in elementary functions for a polynomial $f(x)$ with nonzero degree.

Proof. By our corollary, we have that this is true if and only if there is a $r(x) \in \mathbb{C}(x)$ such that $\frac{1}{f(x)} = r'(x) + r(x)$. Let $r(x) = \frac{p(x)}{q(x)}$ for relatively prime $p(x), q(x)$ with $q(x)$ a monic polynomial. Note that $q(x)$ must be nonconstant since there is obviously no polynomial $r(x)$ solution. We thus want to show there are no $p(x), q(x)$ such that

$$\frac{1}{f(x)} = \frac{p(x)}{q(x)} - \frac{p'(x)q(x) - p(x)q'(x)}{q(x)^2} = \frac{p(x)q(x) + p'(x)q(x) - p(x)q'(x)}{q(x)^2}.$$

This is equivalent to showing that $p(x)q(x) + p'(x)q(x) - p(x)q'(x)$ does not divide $q(x)^2$. Let $q(x) = \prod_{i=1}^{n}(x - r_i)^{e_i}$. From Lemma 5.1, this means $q'(x) = a(x)\prod_{i=1}^{n}(x - r_i)^{e_i - 1}$ for some $a(x)$ relatively prime to $q(x)$. Thus, the quotient simplifies to

$$\frac{p(x)q(x)+p'(x)q(x)-p(x)q'(x)}{q(x)^2} = \frac{(p(x)+p'(x))\prod_{i=1}^{n}(x-r_i)^{e_i} - p(x)a(x)\prod_{i=1}^{n}(x-r_i)^{e_i-1}}{\prod_{i=1}^{n}(x-r_i)^{2e_i}}$$
$$= \frac{(p(x)+p'(x))\prod_{i=1}^{n}(x-r_i) - p(x)a(x)}{\prod_{i=1}^{n}(x-r_i)^{e_i+1}}$$

Let the polynomial in the numerator be $b(x)$. Since $\deg(p') < \deg(p)$ and $\deg(a) < \deg(q)$, $p(x)\prod_{i=1}^{n}(x-r_i)$ has greater degree than the other two terms in $b(x)$. Thus, $b(x)$ isn't constant. For any root r_i of $q(x)$, we have $b(r_i) = -p(r_i)a(r_i) \neq 0$ as both $p(x), a(x)$ are relatively prime to $q(x)$. It follows that $b(x)$ has no common roots with $q(x)$, so $b(x)$ is relatively prime to the denominator. This means $p(x)q(x)+p'(x)q(x)-p(x)q'(x)$ does not divide $q(x)^2$, as desired. □

As a specific example, we have that e^x/x is not integrable in elementary terms.

Example. $\frac{1}{\log(x)}$ is not integrable in elementary functions.

Proof. We show $\int \frac{1}{\log(x)}$ is not elementary. Let $u = \log(x)$. Then $x = e^u$, and $dx = e^u du$. We then have $\int \frac{1}{\log(x)}dx = \int \frac{e^u}{u}du$. From the preceding example, we know this is not elementary. □

Example. e^{e^x} is not integrable in elementary functions.

Proof. Let $u = e^x$, so $du = e^x dx = udx$, or $dx = \frac{du}{u}$. Thus, $\int e^{e^x}dx = \int \frac{e^u}{u}du$, which is again not elementary. □

Example. $\log(\log(x))$ is not integrable in elementary functions.

Proof. We use integration by parts. Let $u = \log(\log(x))$ and $dv = dx$, so $du = \frac{1}{x\log(x)}dx$ and $v = x$. We then have $\int \log(\log(x))dx = x\log(\log(x)) - \int \frac{x}{x\log(x)}dx = x\log(\log(x)) - \int \frac{1}{\log(x)}dx$, but we already know that this integral is not elementary. □

Example. $\frac{\sin x}{x}$ is not integrable in elementary functions.

Proof. Make the substitution $z = ix$, so $dz = i\,dx$. Using the fact that $\sin x = \frac{e^{ix}-e^{-ix}}{2i} = \frac{e^u - e^{-u}}{2i}$, it is then equivalent to show that $\int \frac{e^z - e^{-z}}{z}dz$ is not elementary. Consider the differential field $\mathbb{C}(z,t)$, where $t = e^z$. If this integral is elementary, then Liouville's Theorem tells us that there exist $c_1,\ldots,c_n \in \mathbb{C}, u_1,\ldots,u_n, v \in \mathbb{C}(z,t)$ such that

$$\frac{t^2-1}{tz} = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v'.$$

Let $\mathbb{F} = \mathbb{C}(z)$, so that $u_i, v \in \mathbb{F}[t]$. We may assume that the u_i's not in $\mathbb{F}$ are distinct monic irreducible polynomials of $\mathbb{F}[t]$. Again, write v in its partial fraction decomposition form. Using Lemma 4.1 and reasoning identical to that given in the proof of Liouville's, the only u_i not in $\mathbb{F}$

would be t, so that $\sum_{i=1}^n c_i \frac{u_i'}{u_i} \in \mathbb{F}$. The only possible monic irreducible factor of a denominator in v is t. Writing $v = \sum_j b_j t^j$ and letting $q = \sum_{i=1}^n c_i \frac{u_i'}{u_i}$ like in the proof of Corallary 4.4 yields

$$\frac{t^2-1}{tz} = \Big(\sum_j b_j t^j\Big)' + q = \sum_j (b_j' t^j + j b_j t' t^{j-1}) + q = \sum_j (b_j' t^j + j b_j g' t^j) + q, \text{ or}$$
$$\frac{t^2-1}{z} = \sum_j (b_j' t^{j+1} + j b_j g' t^{j+1}) + qt.$$

Again, both the left-hand side and right-hand side are now polynomials in t. Equating the coefficient of t^2 gives us $\frac{1}{z} = b_1' + b_1$, which by a preceding example is not possible. Thus, $\int \frac{\sin x}{x} dz$ is not elementary.

□

6 Risch's Algorithm

As we have seen in the examples from the previous section, the Integrability Criterion resulting from Liouville's Theorem helps show that many elementary functions do not have an elementary integral. The general problem of elementary integration is whether given any elementary complex function of a single variable, there exists an elementary integral of this function, and if so, how to construct it. Risch's algorithm provides a constructive solution to this general problem. The algorithm looks for an integral of the form given in Liouville's Theorem.

Given an elementary function $f(x)$, Risch's algorithm first constructs a field K containing all the constants of f, then the rational function field $K(x)$, and finally builds a tower $L = K(x)(g_1, \cdots, g_m)$, where the g_i are all the elementary functions needed to express f.

Example. To find $\int \cos x dx$, we first rewrite $\cos x$ as $\frac{e^{ix}+e^{-ix}}{2}$, so the constant field is $K = \mathbb{Q}(i)$, and the tower is $K(x)(g)$, where $g = e^{ix}$. The integrand then becomes $(g^2+1)/(2g)$, which is a rational function in g over the field $F = K(x)$.

Once the tower is found such that $f \in K(x)(g_1, g_2, \cdots, g_m)$, then there are four cases:

1. $m = 0$, the base case, so $f \in K(x)$ is just a rational function of x.
2. g_m is transcendental and logarithmic over $F = K(x)(g_1, g_2, \cdots, g_{m-1})$.
3. g_m is transcendental and exponential over $F = K(x)(g_1, g_2, \cdots, g_{m-1})$.
4. g_m is algebraic over $F = K(x)(g_1, g_2, \cdots, g_{m-1})$.

Risch resolved each of these cases separately. The base case is more straightforward, so we outline it here:

For the base case, take a partial fraction decomposition of $f = P + A/D$, where $P, A, D \in K[x]$, and $\deg(A) < \deg(D)$. Let $D = \prod P_i^{e_i}$ be a prime factorization of D over K. Then further decomposing A/D yields:

$$\int f = \int P + \int \left(\sum_{i=1}^n \sum_{j=1}^n \frac{A_{ij}}{P_i^j} \right)$$

For each i, j, we can find $B_{ij}, C_{ij} \in \mathbb{K}[x]$ using the extended Euclidean Algorithm such that

$$B_{ij}P_i' + C_{ij}P_i = \frac{A_{ij}}{1-j}.$$

Integrating by parts gives us

$$\int \frac{A_{ij}}{P_i^j} = \int \frac{B_{ij}(1-j)P_i'}{P_i^j} + \int \frac{C_{ij}(1-j)P_i}{P_i^j} = \frac{B_{ij}}{P_i^{j-1}} - \int \frac{B_{ij}'}{P_i^{j-1}} + \int \frac{(1-j)C_{ij}}{P_i^{j-1}}$$

Thus, repeatedly performing this reduction, we are left with integrals of the form $\int \frac{E_i}{P_i}$, where the P_i are irreducible, and $\deg(E_i) < \deg(P_i)$. Then we can compute a factorization of P_i over the algebraic closure of K, ie. $P_i = \prod(x - a_{ij})$, and use this factorization to decompose each of the $\frac{E_i}{P_i}$ terms. We thus have

$$\int \frac{E_i}{P_i} = \int \sum_j \frac{b_{ij}}{x - a_{ij}} = \sum_j b_{ij} \log(x - a_{ij}),$$

which is elementary, so this completes the base case of the algorithm.

Similar methods and more advanced techniques can be used to find explicit forms for the logarithmic, exponential, and algebraic cases as well. For the logarithmic and and exponential cases, the same idea is to write a partial fraction decomposition for f, ie. $f = P + A/D$, but in these cases further work is needed to show that $\int P$ is elementary. Liouville's Theorem is used in the penultimate step of exponential and logarithmic cases to show that either an elementary integral does not exist for P (and thus f) or, otherwise, the construction gives a valid elementary antiderivative. The algebraic case is notably more difficult to resolve, and requires knowledge of algebraic curves.

Risch's algorithm has been steadily improved by computer scientists and mathematicians. It is a milestone in computational mathematics. Risch Algorithm forms a basic framework of symbolic integration in modern calculators like Wolfram.

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