THE RUBIK'S CUBE AND GROUP THEORY

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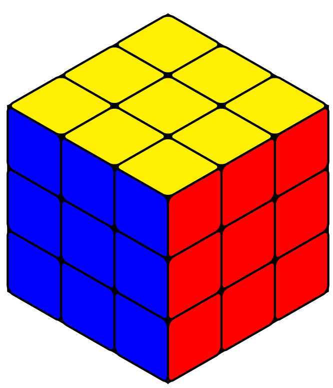
1. Introduction

The Rubik's Cube is a popular children's toy that came onto the market over 40 years ago. The goal of the toy is to scramble the colorful, smaller cubes in any order and try to obtain the original permutation where each side is full of only one color. Although it was not officially marketed as a puzzle until 1975, the cube's creator, Erno Rubik, took over a month to solve the cube for the very first time. Now it has created a booming competition community across the world who go head-to-head in order to see who can solve the Cube the fastest. Currently, the world record is held by Feliks Zemdegs who solved the Cube in 4.22 seconds.

In this paper we will discuss how the Rubik's Cube can be represented as a mathematical concept using group theory and the relation between the possible permutations of the colors on the Cube versus the order of group created the constraints of the Rubik's Cube. The Rubik's Cube refers to the standard, 3x3, unmarked Rubik's Cube where the color on the top and bottom remain the same (yellow stay on top and white stays on the bottom throughout the permutations). The Rubik's cube will be defined as the following:

Definition 1.1. The Rubik's Cube is defined with having six center cubes with one sticker that do not move relative to each other, twelve edge cubes with two stickers, and eight corner cubes with three stickers.

The small cubes in the definition will be now referred to as "cubies" which shockingly is their official name. The entire cube consists of 27 cubies in addition to the pivot point in the middle of the cube. This center pivot point will remain generally irrelevant for the duration of this paper besides the fact that it holds the six center cubies in place.



2. Legal Moves and Positions

In the Rubik's Cube, the moves that are able to be conducted are broken up into two sections: legal and illegal.

Definition 2.1. An illegal move means that the function cannot be carried out independently, i.e. another move if forced to be performed in order for the illegal move to be carried out. This includes flipping a single edge, permuting two cubes, and rotating a single corner.

Definition 2.2. Legal moves mean that the function can be carried out independently, i.e. another move does not need to be performed in order for the legal moved to be carried out.

There are six legal moves that are able to used on the Rubik's Cube.

- F = A 90 degree rotation clockwise of the front face
- B = A 90 degree rotation clockwise of the back face
- U = A 90 degree rotation clockwise of the top face
- D = A 90 degree rotation clockwise of the bottom face
- R = A 90 degree rotation clockwise of the right face
- L = A 90 degree rotation clockwise of the left face

If we wanted to have the face move in the opposite direction, counterclockwise, we would re-write the operation as the prime of said move. It would look something like this:

- F' = A 90 degree rotation counter-clockwise of the front face
- B' = A 90 degree rotation counter-clockwise of the back face
- U' = A 90 degree rotation counter-clockwise of the top face
- D' = A 90 degree rotation counter-clockwise of the bottom face
- R' = A 90 degree rotation counter-clockwise of the right face
- L' = A 90 degree rotation counter-clockwise of the left face

If we wanted to have the face move two times, or 180 degrees, we would re-write the operation as the square of said move. Whether or not the move is prime or not is unimportant as the face will still do a 180 degree rotation. For sake of notation, we will use the clockwise rotation. The 180 degree moves notation would look something life this:

- F^2 = An 180 degree rotation clockwise of the front face
- B^2 = An 180 degree rotation clockwise of the back face
- U^2 = An 180 degree rotation clockwise of the top face
- D^2 = An 90 degree rotation clockwise of the bottom face
- R^2 = An 90 degree rotation clockwise of the right face
- L^2 = An 90 degree rotation clockwise of the left face

These moves a commonly represented using permutation notation (ex. (FUB)(DR)). Legal moves are the only turns allowed in many competitions. The use of illegal moves can result in disqualification and a life-long ban from many competitions. Any set of legal moves create a legal position.

Definition 2.3. A legal position is any reachable permutation of the Rubik's Cube that is able to be reached from the solved Rubik's Cube using a sequence of legal moves.

From the definitions above, it can be determined that the set of these legal positions forms a group. This is because a legal position can be defined with any sequence of legal moves. The binary operator for the group, which we will define as $\mathcal{G}$ for now on, made by the legal moves is defined as $*$. However, it is typically omitted when defining the sequence of moves. This is similar to the interpretation that $f * g = fg$. Now that the composition and operation is identified, it is easy to check that the condition of closure and associativity are able to fulfilled as conditions of a group. Identity can be proven by having an empty set of moves. This also known as a "solved" cube and the legal position that is created by doing no moves.

Now for the fun part, proving the set of legal positions as inverses. We can make an assumption that the group will have an inverse as there are algorithms that have already been determined in order to solve a Rubik's Cube.

Definition 2.4. Let's say that for every legal move P, we can determine that $P^4 = Id$. From this, we can determine that $P^3 = P^{-1}$. This means that for every legal position, there exists an inverse because the legal moves generate the legal positions.

Now that all of these constraints have been met, we can say with full confidence that $\mathcal{G}$ can be defined as a group.

Let's think about this from a more reason based view point rather than a mathematics one. Why would a children's toy be a group? In order to think about this, we must look at what the Rubik's Cube was before it became a cube, a 2-D puzzle. Originating in China in the Yuan Dynasty, the Jiugong Map is a zero-dimensional third-ordered cube. This is based off any n-order magic square that is made from the combination of numbers from $1 - n^2$. The game version of this is simply pacing eight movable pieces in a 3 x 3 square (think of a tic-tac-toe board) and moving the pieces around to create another pattern. This is similar to a slide puzzle, another product of group theory! During the Qing Dynasty, the idea came to expand the dimension of the puzzle from two to three. This puzzle, however, did not rotate until Rubik created his puzzle. It is easy to see conceptually why the slide puzzle is a group. It has closure quite literally, there are only so many places each piece can go. Inverses are just moving the piece in the opposite direction. Identity is the "solved" or original configuration of the puzzle. And associativity is easily proven after doing a few different moves on the puzzle. Once the puzzle rotated, all of these ideas carried over, just on a third axis.

3. The Permutations of the Rubik's Cube versus the Order of Group $\mathcal{G}$

Now that $\mathcal{G}$ has been defined properly as a group, there is now the question of how large is the group of legal positions and is it equal to the number of possible permutations? Let's begin with looking at the permutations.

3.1. Permutations.

Definition 3.1. The number of possible permutations of the Rubik's Cube can be defined as:

$$\frac{(8!\cdot 3^8 \cdot 12! \cdot 2^{12})}{(3\cdot 4)} = 4.3252 \cdot 10^{19}$$

This is because there are 8 corner pieces that can be arranged in 8! ways. Each of these arrangements in turn has 3 orientations, meaning there are 3^8 permutations for the corner cubes. There are then 12 edge pieces that can be arranged in 12! ways. Each of these arrangements in turn has 2 orientations, meaning there are 2^{12} permutations for the edge cubes. However, in order to make sure there are no illegal moves being made, we must account for how many of these arrangements are possible. For the corner cubes, only 1/3 of the permutations are possible. For the edge cubes, only 1/4 of the permutations are possible. This gives us the possible permutations with the equation shown above.

In this equation, however, there is no distinction between the legal and non-legal moves that are used to obtain these permutation. For all we know, the user could peel off the stickers and rearrange them manually. If we take the non-legal positions out of the equation, we are able to get the order of $\mathcal{G}$. So, how do we do this mathematically?

3.2. **The Order.**

3.2.1. *Reachable Permutations.* Well, it is now an important time to note that every legal position can only be represented by an even number of legal moves. In order to understand this concept fully, it must be realized that an individual legal move will always perform an even number of swaps.

Definition 3.2. A swap will be defined as the exchange between two location on the Rubik's Cube.

What this means for us is that every permutation that is achieved by an odd number of legal moves in considered an illegal position.

Definition 3.3. An illegal position will be defined as any position that is unattainable from using legal moves. At least one illegal move must be used in order for the position to be achieved, hence the title.

Now we know that at least half of the permutations calculated earlier will be unattainable given our constraints of legality.

3.2.2. *Edge Orientations.* After some more investigation, we can discover that each legal move will also flip an even number of edges only.

Definition 3.4. A flip is very similar to a swap, but for a set of cubies instead of a single cubie.

In order to prove this observation, we must establish that an edge in an incorrect position, but has correct orientation if when it is moved to its correct position using the left, right, top and bottom faces, it would have the correct orientation. It is quite easy to see that these moves will consistently flip zero edges, an even number. Now, the only remaining faces to deal with are the front and back. Any moves for these faces (F and B) will flip all four of the cubes edges, another even number. Given all of this information, we can determine that it is impossible to flip an odd number of edges using legal moves. This means that another half of the orientations calculated will be unattainable given our constraints of legality.

3.2.3. *Corner Orientations.* We have dealt with the permutations and edge orientations, but we are not done in excluding illegal moves. We still must consider the corner orientations from the $4.3252 \cdot 10^{19}$ permutations. These are slightly more difficult to determine how many are legal versus illegal as there are three possible after some observations of the Rubik's Cube, it becomes apparent that each legal move will twist the corners in a way that the sum of their orientations is divisible by 3. So, how do we prove this mathematically?

In order to begin this proof, we must make another observation about the cube: each corner cubie will always have at least one of its faces on the top or bottom face of the Rubik's Cube. This is to say that a "correct" corner orientation will be met if its top or bottom sticker is facing up or down. Using this observation, it is clear to see that an illegal twist of the top and bottom faces does not change the orientation of the corner cubies. Therefore, the total sum of the orientations will become 3 regardless of the amount of legal moves made.

3.2.4. *Order/Permutations.* All of these conclusions combined with the observation that there is a 1 in 12 chance of obtaining a solvable Rubik's Cube from fully disassembling the cubies and reassembling them in a random order, we can determine that $1/12(1/2 \cdot 1/2 \cdot 1/3)$ or $1/144$ of the permutations of the cube are obtainable using only legal moves.

So out of the possible $4.3252 \cdot 10^{19}$ permutations of the cube, $3.004 \cdot 10^{17}$ are obtainable.

3.3. **Relation.** All of these exclusions begs the question of why is there this discrepancy between the order of group $\mathcal{G}$ and the amount of possible permutations. For starters, it excludes the permutations that are only reachable by removing the stickers and replacing them in their designated area. This is where we got the $1/12$ and the first $1/2$ in the equation for the order of group $\mathcal{G}$. In any competition, this would quickly get you disqualified regardless of how much time it would take to peel off all 54 stickers and reattach them firmly. Then the matter of actual illegal moves factors in such as twisting a cubie in order to change the entire cube's orientation. For the edge cubies, this is only able to be done twice as there are only two faces to their cubie. However for corner cubies, this can be done three time because there are three possible permutations for each corner cubie if illegal moves are taken into account. Only one of these orientations for both types of cubies are able to be obtained through legal moves. This is where we get the $1/2$ and $1/2$. It is through the combinations of all of these constraints that we end up with $1/144$ of the permutations that are able to be obtained using only legal moves.

4. SUBGROUPS

There are a few obvious subgroups that emerge from the Rubik's Cube. The most notable of these are the corner group and the edge group. These are both interesting subgroups to study due to the constraints they both have. The corner group consists of all the corner cubies. These have three visible faces at all times and only have eight possible places they can go to, the corners. The edge group consists of all the edge cubies. These have only two visible faces at a time, but have 12 possible places they can go. The composition and operation identified in earlier still hold for both of these supposed subgroups. Therefore, the condition of closure and associativity are able to be checked and identified quite easily in order to fulfill the conditions of a group. The identity hold up as well, they cubies for each subgroup stay in their "resting" positions relative to a solved cube. The proof for identifying closure in both of these subgroups will be left as an exercise for the reader. Both of these subgroups become very important when solving the Rubik's Cube as if it were a permutation.

5. SOLVING

In the last 10 years, it has been discovered that given any of the quadrillion orientations of a cube, the user is never more than 20 moves off from a solved cube. This is commonly referred to as "Gods Number" because, let's be honest, it is usually an act of God to solve the Cube unless group theory is applied. Using the notation demonstrated above, the permutation notation is used to show the moves needed to be completed in order to go back to the resting/solved position. While there is indeed a set of moves that we will go over below, there is not a certain number of times that they must be carried out. For instance, a Right Corner switch is used to get a corner cubie into place. However, this series of moves may need to be carried out anywhere from one to five times in order to get the demonstrated result.

This is because there is no way to accurately predict the position of all faces of the cubies. There are simply too many permutations of the Rubik's Cube in order for it estimated in any way mathematically. This means that the user must hold some sort of intuition to solve the cube, frustrating many common folk (Ohioans) on their in-ability to solve the Rubik's Cube just because they do not want to have to think. Here are the steps for solving the Rubik's Cube.

- **The Flower:** The user first orients the cube with the yellow center cubie facing the sky and the white center cubie facing the ground. This will be standard for the rest of the solve. The user then must manually create the white "flower" aka get the four white edge cubies surrounding the white center cubie on the bottom. This can be the most frustrating step for many people, mainly because it is the only one they will take.
- **The Corners:** Next the user will then locate one of the four white corner cubies and get it in the upper corner corresponding with the colors on either side (i. e. if the cubie has white, green, and orange faces, the user will place the cube in the corner between the orange and green center cubies). Then depending on the location of the corner cubie in the users hand, there will be either a left or right hand corner switch. If the cubie is to the left side of the center, a left switch will be preformed. If the cubie is to the right side of the center, a right switch will be preformed.

  Definition 5.1. A left hand switch consists of the moves: $(U'L'UL)$.

  Definition 5.2. A right hand switch consists of the moves: $(URU'R')$.

  The user will do this until the bottom face is covered with white cubies.
- **The Edges:**The user will now focus on the cubies above the center cubies of the side faces (not the yellow and white center cubies). Match the color of the cubie on the top row with the color of the center cubie. Then determine if the opposite color on top is to the left or the right of the cubie in question. Preform the respective switch described above for the left or the right (i.e. if the color is on the left, perform a let switch). The user will then notice that a white corner cubie has been moved out of place, this can be fixed by preforming the steps in **The Corners**. These steps should be repeated until the bottom two layers of the Cube are "solved".
- **The Top Layer Edges:** Now for a more technical view overall. The user should realize that there is still only one yellow cubie on top. In order to get two yellow edge cubies next to it, the user will perform the moves $(FURU'R'F')$. If the result desired is not accomplished, repeat the moves until it is fulfilled. Orient the cube so that the line made by the yellow cubies are horizontal. The user will then repeat these moves until they achieve an L shape and then a full cross of yellow cubies. There is no telling how many times the user will need to perform these set of moves as it is dependent on the orientation of the Cube.
- **The Top Layer Corner:** Once a cross has been achieved, orient the Cube so that a yellow cubie is on the bottom left from a birds eye view. Now the user will complete a series of moves that is referred to as a "Right Sune."

  Definition 5.3. A Right Sune consists of the moves: $(RUR'URU^2R')$.

  The user will repeat a Right Sune until they get the entire top layer full of yellow cubies, regardless of the top layers edge cubies orientation. Once again, there is no telling how many times the user will need to complete this set of moves.

- **Top Layer Corner Orientations:** The user will now orient the Cube so that the cubies on the front and back on the right side are the same color. No moves should be involved, just the rotation of the Cube in the users hand while keeping the yellow cubies facing up. Once this orientation has been achieved, the user will complete the following set of moves until all of the corner cubies are in their solved position. The moves are as follows: $(L'URU'LU^2R'URU^2R')$.
- **Top Layer Edge Orientations:** If possible, the user should place the orange cubies on the backside of the cube. Now, the user will repeat the following moves until the Cube has reached a solved position, although it usually only has to be carried out once. The moves are as follows: $(RUR'URU^2R'UL'U'LU'L'U^2LU')$.

If these set of moves do not result in a solved cube, it can be assumed that the user did not follow the instructions or made an incorrect move at some point.

6. Conclusion

It is with these conclusions that we are able to see that with group theory, not everything is as it seems at first glance. The Rubik's Cube is known for it's "billion" different possibilities that has kids all around the world craving to solve the puzzle. However, many of them are not obtainable for legal moves. Only 1/12 of the possible permutations are obtainable without removing the stickers. It is the fascinating mathematics behind the Rubik's Cube and its applications in group theory that allow it to be the timeless toy we have all become to love.

References

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