A GENTLE INTRODUCTION TO LIE GROUPS

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Abstract. Lie theory was developed by mathematician Sophus Lie in the late 19^{th} century. In essence it rests on the theory of continuous groups or groups with a continuous operation. In this paper, we will introduce some preliminary knowledge about manifolds in the first section of this paper, and then specifically discuss Lie groups emerged from it.

1. A Brief Introduction to Manifolds

Surface, one of the oldest objects geometricians have been studying, can be called a 2d *manifold* in the field of modern differential geometry. But, what is a manifold?

A manifold is a topological space that is "locally" resembles the Euclidean space ($\mathbb{R}^n$). One way to think of it is by taking a geometric object from $\mathbb{R}^k$ and trying to "fit" it into $\mathbb{R}^n, n > k$.

Of course, there is a precise definition of the manifold in topology—a manifold is defined as a set that is homeomorphic to Euclidean space. A homeomorphism is a continuous one-to-one mapping that preserves topological properties.

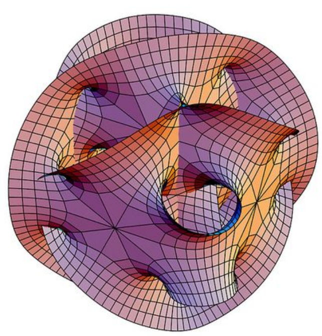

Figure 1. Calabi-Yau Manifold.

Figure 1 is an illustration the Calabi-Yau Manifold. We can observe that each point on this "surface" looks like (or homeomorphic to) a $\mathbb{R}^2$ space, but in fact, this kind of manifold holds much more sophisticated algebraic properties which we will not discuss here.

Let us now take a look at the formal formal definition of a topological manifold.

Definition 1.1. An n-dimensional topological manifold M is a topological Hausdorff space with a countable base which is locally homeomorphic to $\mathbb{R}^n$. This means that for every point p in M there is an open neighbourhood U of p and a homeomorphism $\varphi : U \to V$ which maps the set U onto an open set $V \subset \mathbb{R}^n$. Additionally:

Date: June 8, 2020.

- The mapping $\varphi : U \to V$ is called a chart or coordinate system.
- The set U is the domain or local coordinate neighbuorhood of the chart.
- The image of the point $p \in U$, denoted by $\varphi(p) \in \mathbb{R}^n$, is called the coordinates or local coordinates of p in the chart.
- A set of charts, $\{\varphi_\alpha | \alpha \in \mathbb{N}\}$, with domains U_α is called the atlas of M, if $\bigcup_{\alpha\in\mathbb{N}} U_\alpha = M$.

This definition can be difficult to understand since many necessary related definitions have not been introduced. We will not be focusing on the topology in this paper, so that is not essential to our discussion. Let us take a look at this figure below to clarify some ideas.

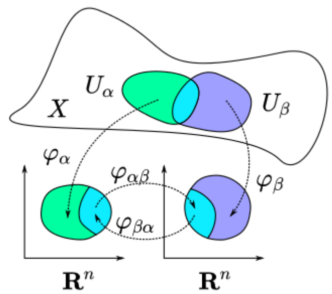

Figure 2. Two intersecting patches (green and purple with cyan/teal as the intersection) on a manifold with different charts (continuous 1-1 mappings) to $\mathbb{R}^n$ Euclidean space. Notice that the intersection of the patches have a smooth 1-1 mapping in $\mathbb{R}^n$ Euclidean space, making it a differential manifold.

Figure 2 also has another mapping between the intersecting parts of U_α and U_β in their respective chart coordinates called a transition map, given by $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ and $\phi_{\beta\alpha} = \varphi_\alpha \circ \varphi_\beta^{-1}$ (their domain is restricted to either $\varphi_\alpha (U_\alpha \cap U_\beta)$ or $\varphi_\beta (U_\alpha \cap U_\beta)$, respectively.

These transition functions are vital because depending on their differentiability, they define a new class of differentiable manifolds (denoted by C^k if they are k-times continuously differentiable). The most important one is infinitely differentiable, which is called smooth manifolds.

To actually calculate things such as distance on a manifold, we have to introduce few indispensable concepts: the first one is the tangent space T_xM of a manifold M at a point x. Imagine you are walking along a curve on a smooth manifold, as you pass through the point x you implicitly have velocity (magnitude and direction) that is tangent to the manifold, in other words, a tangent vector. The tangent vectors made in this way from each possible curve passing through point x make up the tangent space at x. For a 2d manifold (embedded in 3d), this would be a plane. Figure 3 shows a visualization of this on a manifold.

Those tangent vectors are "velocities" on a manifold, telling us how they "move". However, they are somehow confined in a local coordinate system. To address this, we need to understand what basis is, which is the second indispensable Recall that a vector has its own coordinates that correspond to particular basis vectors. This is important because we want to be able to do analysis on the manifold between points and charts, not just at a single point/chart. So understanding how the tangent spaces relate between different points (and potentially charts) on a manifold is important.

To understand how to construct the tangent space basis, let's first define an arbitrary function $f : M \to \mathbb{R}$ and assume we still have our good old smooth parametric curve

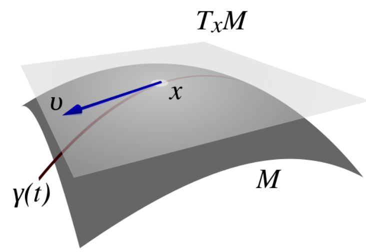

Figure 3. A tangent space T_xM for manifold M with tangent vector $\mathbf{v} \in T_xM$, along a curve traveling through $x \in M$.

$\gamma : t \to M$. Now we want to look at a new definition of "velocity" relative to this test function: $\left.\frac{df\circ(t)}{dt}\right|_{t=t_0}$ at our point on the manifold p. Basically the rate of change of our function f as we walk along this curve.

However, we can do a "trick" by introducing a chart (φ) and its inverse (φ^{-1}) into this measure of "velocity":

$$\begin{aligned}
\left.\frac{df\circ\gamma(t)}{dt}\right|_{t=t_0} &= \left.\frac{d\left(f\circ\varphi^{-1}\circ\varphi\circ\gamma\right)(t)}{dt}\right|_{t=t_0} \\
&= \left.\frac{d\left(\left(f\circ\varphi^{-1}\right)\circ(\varphi\circ\gamma)\right)(t)}{dt}\right|_{t=t_0} \\
&= \sum_i \left.\frac{\partial\left(f\circ\varphi^{-1}\right)(x)}{\partial x_i}\right|_{x=\varphi^o\gamma(t_0)} \left.\frac{d(\varphi\circ\gamma)^i(t)}{dt}\right|_{t=t_0} \\
&= \sum_i \left.\frac{\partial\left(f\circ\varphi^{-1}\right)(x)}{\partial x_i}\right|_{x=\varphi(p)} \left.\frac{d(\varphi\circ\gamma)^i(t)}{dt}\right|_{t=t_0} \\
&= \sum_i (\text{basis for component } i).
\end{aligned}$$

Note that the third line is just an application of the multi-variable calculus chain rule. We're going to take the basis and re-write it:

$$\left(\frac{\partial}{\partial x^i}\right)_p (f) := \frac{\partial\left(f\circ\varphi^{-1}\right)(\varphi(p))}{\partial x_i},$$

which simply defines some new notation for the basis. There is a convention that φ is implicitly specified by x^i. So if we have some other chart, say, ϑ, then we label its local coordinates with y^i. But it's important to remember that when we're using this notation implicitly there is a chart behind it.

There's another thing we need to look at: what is f ? We know it's some test function that we used, but it was arbitrary. And in fact, it's so arbitrary we're going to get rid of it! So we're just going to define the basis in terms of the operator that acts of f and not the actual resultant vector! So every tangent vector $v \in T_pM$, we have:

$$\begin{aligned}\mathbf{v} &= \sum_{i=1}^{n} v\left(x^{i}\right) \cdot \left(\frac{\partial}{\partial x^{i}}\right)_{p} \\ &= \sum_{i=1}^{n} \left.\frac{d(\varphi \circ \gamma)^{i}(t)}{dt}\right|_{t=t_0} \cdot \left(\frac{\partial}{\partial x^{i}}\right)_{p},\end{aligned}$$

which suggest the basis is actually a set of differential operators.

2. Transition to Lie Groups

In this section we will precisely define Lie groups, and Lie algebras. One of the reasons that Lie groups are nice is that they have a differential structure (or C^{∞}), which means that the notion of tangent space makes sense at any point of the group.

There are numerous theorems, lemmas, and propositions in Lie groups. To have a overall look of what Lie groups and Lie algebra are, we will introduce some of them without proofs, as they are too advanced for this introductory article.

First, we will introduce the *smooth manifold*.

The intuition behind the notion of a smooth manifold in $\mathbb{R}^N$ is that a subspace M is a manifold of dimension m if every point $p \in M$ is contained in some open subset U of M (in the subspace topology that can be parametrized by some function $\varphi : \Omega \to U$ from some open subset Ω in $\mathbb{R}^m$ containing the origin, and that φ has some nice properties that allow the definition of smooth functions on M and of the tangent space at p. For this, φ has to be at least a homeomorphism, but more is needed: φ must be smooth, and the derivative $\varphi'\left(0_m\right)$ at the origin must be injective (letting $0_m = \underbrace{(0, \ldots, 0)}_{m}$).

Definition 2.1. Given any integers N, m, with $N \geq m \geq 1$, an *m-dimensional smooth manifold* in $\mathbb{R}^N$, for short a manifold, is a nonempty subset M of $\mathbb{R}^N$ such that for every point $p \in M$ there are two open subsets $\Omega \subseteq \mathbb{R}^m$ and $U \subseteq M$, with $p \in U$, and a smooth function $\varphi : \Omega \to \mathbb{R}^N$ such that φ is a homeomorphism between Ω and $U = \varphi(\Omega)$, and $\varphi'\left(t_0\right)$ is injective, where $t_0 = \varphi^{-1}(p)$; see Figure 4.1. The function $\varphi : \Omega \to U$ is called a *(local) parametrization* of M at p. If $0_m \in \Omega$ and $\varphi\left(0_m\right) = p$, we say that $\varphi : \Omega \to U$ is *centered at* p.

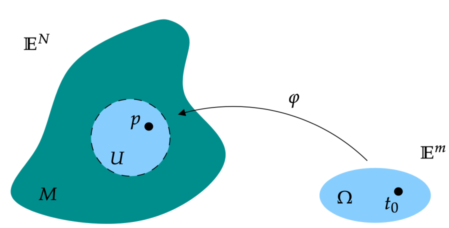

Figure 4. A manifold in $\mathbb{R}^N$.

Notice that $\mathbb{E}^N$ and $\mathbb{E}^m$ refer to Euclidean spaces.

Saying that $\varphi'\left(t_0\right)$ is injective is equivalent to saying that φ is an immersion at t_0. Recall that $M \subseteq \mathbb{R}^N$ is a topological space under the subspace topology, and U is some open subset of M in the subspace topology, which means that $U = M \cap W$ for some open subset W of $\mathbb{R}^N$. since $\varphi : \Omega \to U$ is a homeomorphism, it has an inverse $\varphi^{-1} : U \to \Omega$ that is also a homeomorphism, called a *(local) chart.* since $\Omega \subseteq \mathbb{R}^m$, for every point $p \in M$ and every parametrization $\varphi : \Omega \to U$ of M at p, we have $\varphi^{-1}(p) = (z_1, \ldots, z_m)$ for some $z_i \in \mathbb{R}$ and we call $z_1, \ldots, z_m$ the *local coordinates of* $p\,(w.r.t.\ \varphi^{-1})$. We often refer to a manifold M without explicitly specifying its dimension (the integer m).

In order to have transit to Lie algebra, let's now rigorously define some terms as to manifolds.

Lemma 2.2. *Given an m-dimensional manifold M in $\mathbb{R}^N$, for every $p \in M$ there are two open sets $O, W \subseteq \mathbb{R}^N$ with $0_N \in O$ and $p \in M \cap W$, and a smooth diffeomorphism $\varphi : O \to W$, such that $\varphi(0_N) = p$ and*

$$\varphi\left(O \cap \left(\mathbb{R}^m \times \{0_{N-m}\}\right)\right) = M \cap W.$$

There is an open subset Ω of $\mathbb{R}^m$ such that

$$O \cap \left(\mathbb{R}^m \times \{0_{N-m}\}\right) = \Omega \times \{0_{N-m}\},$$

and the map $\psi : \Omega \to \mathbb{R}^N$ given by

$$\psi(x) = \varphi\left(x, 0_{N-m}\right)$$

is an immersion and a homeomorphism onto $U = W \cap M$; so ψ is a parametrization of M at p. We can think of φ as a promoted version of ψ which is actually a diffeomorphism between open subsets of $\mathbb{R}^N$; see Figure 5.

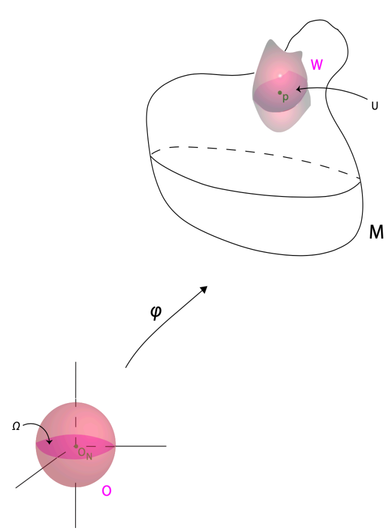

Figure 5. An illustration of Lemma 2.2, where M is a surface embedded in $\mathbb{R}^3$, namel $m = 2$ and $N = 3$.

Proposition 2.3. *A subset $M \subseteq \mathbb{R}^{m+k}$ is an m-dimensional manifold iff either*

(1) *For every $p \in M$, there is some open subset $W \subseteq \mathbb{R}^{m+k}$ with $p \in W$, and a (smooth) submersion $f : W \to \mathbb{R}^k$, so that $W \cap M = f^{-1}(0)$,*

or

(2) *For every* $p \in M$, *there is some open subset* $W \subseteq \mathbb{R}^{m+k}$ *with* $p \in W$, *and a (smooth)* map $f : W \to \mathbb{R}^k$, *so that* $f'(p)$ *is surjective and* $W \cap M = f^{-1}(0)$.

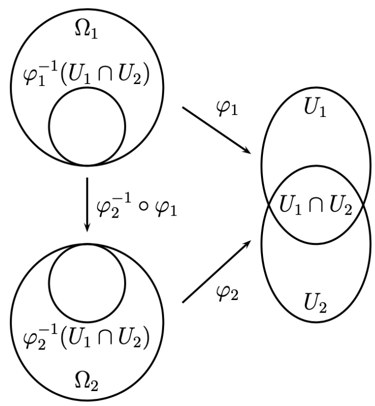

Figure 6. An illustration of parameterization and transition functions.

Now, we will start introducing some linear Lie groups. Let us begin with reviewing the definition of a group.

Definition 2.4. A group is a set G together with a binary operation $\cdot : G \times G \to G$ (meaning that for any $g, h \in G$, we have a well-defined element $g \cdot h \in G$, sometimes also simply written gh), with the following properties:

- There is an element $e \in G$ so that $eg = ge = g$ for all $g \in G$ We call e an identity element of G.
- If $g, h, k \in G$, then $g(hk) = (gh)k$ (associativity).
- For every $g \in G$, there is some element $h \in G$ so that $gh = hg = e$. We call h an inverse of g.

And here is the definition for the Lie group.

Definition 2.5. A Lie group is a nonempty subset G of $\mathbb{R}^N (N \geq 1)$ satisfying the following conditions:

- G is a group.
- G is a manifold in $\mathbb{R}^N$
- The group operation $\cdot : G \times G \to G$ and the inverse map $^{-1} : G \to G$ are smooth.
  (Smooth maps are defined in Definition 4.8). It is immediately verified that $\mathbf{GL}(n, \mathbb{R})$ is a Lie group. since all the Lie groups that we are considering are subgroups of $\mathbf{GL}(n, \mathbb{R})$, the following definition is in order.

Definition 2.6. A linear Lie group is a subgroup G of $\mathbf{GL}(n, \mathbb{R})$ (for some $n \geq 1$) which is a smooth manifold in $\mathbb{R}^{n^2}$

Let $\mathrm{M}_n(\mathbb{R})$ denote the set of all real $n \times n$ matrices (invertible or not). If we recall that the exponential map

$$\exp : A \mapsto e^A$$

is well defined on $\mathrm{M}_n(\mathbb{R})$, we have the following crucial theorem due to Von Neumann and Cartan.

Theorem 2.7. *(Von Neumann and Cartan, 1927) A closed subgroup G of GL(n, R) is a linear Lie group. Furthermore, the set $\mathfrak{g}$ defined such that*

$$\mathfrak{g} = \left\{X \in \mathrm{M}_n(\mathbb{R}) | e^{tX} \in G \text{ for all } t \in \mathbb{R}\right\}$$

is a nontrivial vector space equal to the tangent space $T_I G$ at the identity I, and $\mathfrak{g}$ is closed under the Lie bracket $[-,-]$ defined such that $[A, B] = AB - BA$ for all $A, B \in \mathrm{M}_n(\mathbb{R})$.

Theorem 2.7 applies even when G is a discrete subgroup, but in this case, $\mathfrak{g}$ is trivial (i.e., $\mathfrak{g} = \{0\}$). For example, the set of nonnull reals $\mathbb{R}^* = \mathbb{R} - \{0\} = \mathbf{GL}(1, \mathbb{R})$ is a Lie group under multiplication, and the subgroup

$$H = \{2^n | n \in \mathbb{Z}\}$$

is a discrete subgroup of $\mathbb{R}^*$. Thus, H is a Lie group. On the other hand, the set $\mathbb{Q}^* = \mathbb{Q} - \{0\}$ of nonnull rational numbers is a multiplicative subgroup of $\mathbb{R}^*$, but it is not closed, since $\mathbb{Q}$ is dense in $\mathbb{R}$. Hence $\mathbb{Q}^*$ is not a Lie subgroup of $\mathbf{GL}(1, \mathbb{R})$.

The first step in proving Theorem 2.7is to show that if G is a closed and nondiscrete subgroup of $\mathbf{GL}(n, \mathbb{R})$ and if we define $\mathfrak{g}$ just as $T_I G$ (even though we don't know yet that G is a manifold), then $\mathfrak{g}$ is a vector space satisfying the properties of Theorem 2.7.

Proposition 2.8. *Given any closed subgroup G in $\mathbf{GL}(n, \mathbb{R})$, the set*

$$\mathfrak{g} = \left\{X \in \mathrm{M}_n(\mathbb{R}) | X = \gamma'(0), \gamma : J \to G \text{ is a } C^1 \text{ curve in } \mathrm{M}_n(\mathbb{R}) \text{ such that } \gamma(0) = I\right\}$$

satisfies the following properties:

- $\mathfrak{g}$ *is a vector subspace of* $\mathrm{M}_n(\mathbb{R})$.
- *For every $X \in \mathrm{M}_n(\mathbb{R})$, we have $X \in \mathfrak{g}$ iff $e^{tX} \in G$ for all $t \in \mathbb{R}$.*
- *For every $X \in \mathfrak{g}$ and for every $g \in G$, we have $gXg^{-1} \in \mathfrak{g}$.*
- $\mathfrak{g}$ *is closed under the Lie bracket.*

Proposition 2.9. *Let G be a subgroup of $\mathbf{GL}(n, \mathbb{R})$, and assume that G is closed and not discrete. Then $\dim(\mathfrak{g}) \geq 1$, and the exponential map is a diffeomorphism of a neighborhood of 0 in $\mathfrak{g}$ onto a neighborhood of I in G. Furthermore, there is an open subset $\Omega \subseteq \mathrm{M}_n(\mathbb{R})$ with $0 \in \Omega$, an open subset $W \subseteq \mathbf{GL}(n, \mathbb{R})$ with $I \in W$, and a diffeomorphism $\Phi : \Omega \to W$ such that*

$$\Phi(\Omega \cap \mathfrak{g}) = W \cap G.$$

3. Summary

In this paper we have introduced the concepts of manifold and Lie groups through a gentle introduction and rigorous explanations. Lie groups are remarkably important in both differential geometry and algebra, as it possesses properties of manifolds, which is the cornerstone of differential geometry. We have given several germane definitions, theorems, lemmas, and propositions in the second portion of this paper, for compactness, without proofs. However, it is also important to learn those proofs, if one wants to understand how those theorems are proved with the language of Lie groups and differential geometry completely. I highly recommend reading *Differential Geometry and Lie Groups — A Computational Perspective* by Jean Gallier and Jocelyn Quaintance as an excellent material to study Lie groups and some intersection with differential geometry, to learn the profound meaning of Lie's theory.

References

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