BOOLEAN ALGEBRAS

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CONTENTS

ABSTRACT

In this paper we will learn about Boolean Algebras, and how they pertain to abstract algebra, culminating in the Stone Representation Theorem.

INTRODUCTION

A sign of good mathematics is when two unrelated areas are fundamentally connected. Although it may seem that Abstract Algebra and Topology are distinct, there are surprising relations due to Boolean Algebras. Through Stone Spaces, which we will explore in this paper, the study of symbols and the study of geometric objects are closely linked.

SYMBOLS OF LOGIC

Before we can look at Boolean Algebras, we must understand symbolic logic. In that, 1 represents true and 0 represents false. We will start off with 2 operations and($\wedge$) and or($\vee$). I will write their multiplication tables.

$\wedge$	1	0
1	1	0
0	0	0

$\vee$	1	0
1	1	1
0	1	0

In general, we should think that and outputs true when both inputs are true and or outputs true when at least 1 input is true.

We will also introduce the operation not ($\neg$) which inverts the input.We should think of $\wedge$ like addition and $\vee$ like multiplication. When we do, we see properties similar to addition and subtraction. Some properties are

Theorem 1. *Identity:* $x \wedge 0 = x, x \vee 1 = x, x \vee 0 = 0$ *Commutative property :*$x \wedge y = y \wedge x$, $x \vee y = y \vee x$ *Distributive property:* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Additionally, there are some properities are not in all algebras.

Theorem 2. *Idempotence:* $x \wedge x = x$, $x \vee x = x$ *Distributive:* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

The 2 element Boolean Algebra is an Abelian Group.

Definition of A Boolean Algebra

Before we formally define a Boolean Algebra, let us first define two terms which are very important to Boolean Algebras. A **Partially ordered Set**, sometimes shortened to poset. The order comes from a comparison operation, called $\leq$, and it has the properties:

Definition 3.
Reflexivity - Any element can be compared with itself
Transativity - Comparsions are transitive
Symmetry- if $a \leq b$ and $b \leq a$, then a=b.

A poset is partially ordered, as opposed to totally ordered, because we never specified that we can compare any 2 elements.

The next definition we need is that of a

Definition 4. ***lattice**. It is a poset where each element has a unique supremum and infimum.*

Definition 5. *The definition of a Boolean Algebra is distributive lattice. That means it follows the distributive property, which we discussed previously.*

Probably the most famous Boolean Algebra is the set 0,1, along with the Boolean operations. We call this **2**.

Definition 6. *Similarly to other Abstract Algebra structures, Boolean Algebras have homomorphisms, defined as*

$f(a \wedge b) = f(a) \wedge f(b)$, $f(a \vee b) = f(a) \vee f(b)$, *and* $f(\neg a) = \neg f(a)$

Field of Sets

A **Power Set** of a set s, is the set of all subgroups of s.

Definition 7. *If we have 2 sets, called $X, \vec{F}$, then we say that X is a **field of sets**if it is a collection f subsets of F that is closed under the set theory operations of union ($\cup$), intersection ($\cap$) and complement (/)*

Stone space

We will introduce the subject of topological spaces, which may seem random now, but it is in fact linked to Boolean Algebras.

Definition 8. *A **Stone Space** is a topological space that has 3 properties.*

1.It is a Hausdorff space. Shortly said, this means distinct points have distinct neighborhoods. Longly said, a neighborhood is the set of points in a topological space where one can move without leaving the set. If x and y are distinct points, then the neighborhoods of x and y, P,Q are disjoint.

2. It is compact, meaning it is closed and bounded at all points.

3. It is totally disconnected, meaning it has no subsets that cannot be written as the union of 2 disjoint other subsets.

Hausdorff is a bit complicated, so I will give a definition. The real numbers are a Hausdorff space. In fact most "normal" spaces people encounter are Hausdorff, and it's not until more interesting spaces that non-Hausdorff spaces start to appear.

The most famous example of a Stone space is the Cantor Set. I won't prove that the Cantor set is a Stone space, but we will prove that S(A) is a stone space where S(A) is the set of all homomorphisms from a Boolean Algebra A to **2**.

Theorem 9. *S(A) is a Stone Space.*

$\mathbf{2}^A$is homomorphic to $\mathbf{2} \times \mathbf{2} \times ...\mathbf{2}$ for each element of A, because it corresponds to where the function maps each element of A. We now use Tychonoff's Theroem, which states that the product of compact topological spaces is also compact and similarly for Hausdorff. The remaining condition is that it is totally disconnected. An alternate definition of totally disconnected if the subbase (smallest topology contained) is clopen (is both closed and open).

We also need to know that a set is closed if its complement is open.

let f_a be a function in $\mathbf{2}^A$.We will form a subbase with the 2 open sets $f \in \mathbf{2}^A : f_a = 1$ and $f \in \mathbf{2}^A : f_a = 0$/ A set of the first form is the complement of some set of the second form, and vice versa, so every open set in this subbase is clopen.

Stone Representation

Theorem 10. *The **Stone Representation Theorem** states that every Boolean Algebra has a corresponding Stone Space.*

We can prove this, but the proof is a bit hard.

Definition 11. *If X is a Stone space, then the dual algebra of X is the class of clopen sets in X.*

Let A be a Boolean Algebra and let B be the dual algebra of the Stone space. We want an isomorphism between A and B because that will prove the theorem. To make it easier I will give the relation and then we will prove it is an isomorphism. Let f(p) = $x \in S(A)|x(p) = 1$. Recall the definition of a homomorphism for Boolean Algebras. We first should that f(¬p)=$\neg f(p)$. f($\neg p$) = $x|x(\neg p) = 1$. Since x is a homomorphism, we can say $x|x(\neg p) = 1 = x|\neg x(p) = 1 = x|x(p) = 0$. This is the complement of f(p), meaning we have done the first step. Next we want to show that f($p \vee q$)=f(p)∪f(q). f($p \vee q$) = $x|x(p \vee q) = 1 = x|x(p) \vee x(q) = 1 = x|x(p) \cup x|x(q) = f(p)(q)$. The same argument applies for ∩ as well.

We have shown it is a homomorphism, but we want to show it is an isomorphism.

Suppose f(p)=f(q). We want to show that p=q.

$\emptyset = f(p) \cap \neg f(q) = f(p \wedge \neg q)$. This means $p \wedge \neg q = 0$, so p=q.

This completes the proof.

Conclusion

I hope that in this exercise which connections the Boolean Algebras to Topology, you have witnessed how unexpected and elegant mathematics can be.

If you become interested in these fields and would like some further study, I recommend researching how Category Theory represents the Stone Representation Theorem.

References

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