Quaternions

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1 Introduction

1.1 What are Quaternions?

Quaternions are a number system that extends beyond complex numbers. The Irish mathematician William Rowan Hamilton discovered them in 1843 when he was crossing the Brougham Bridge bridge on his way to the Royal Irish Academy. He carved the fundamental formula for quaternions, $i^2 = j^2 = k^2 = ijk = -1$, into the stone. Just as the complex numbers can be used to examine transformations in the 2-Dimensional plane, quaternions deal with transformations in 3-Dimensional space.

1.2 Motivation

Quaternions are useful for calculating 3D rotations, and are used for computer graphics and computer vision. The products of the quaternion units themselves (i, j, and k) can be represented as 90° rotations in 4D space. Computer animations utilize quaternions to rotate objects in 3-Dimensional space.

2 Preliminary Information

Definition 2.1 (Linearly independent). *Given a vector space V and a field F, a subset $X \in V$ is linearly independent if for all $x_i \in X$ and $f_i \in F$, when $f_1x_1 + f_2x_2 + \cdots + f_{n-1}x_{n-1} + f_nx_n = 0$, $f_1 = f_2 = \cdots = f_{n-1} = f_n = 0$.*

We say this because if some $f_i, f_j \neq 0$, then we have $f_ix_i + f_jx_j = 0$ and thus $x_i = -f_jx_j/f_i$. Therefore, x_i is linearly dependent on x_j (note that the same process can be done for any amount of x_is that have an $f_i \neq 0$).

Definition 2.2 (Spanning). *Given a subset X of V over some field F, the subset X is a spanning set if for every vector $v \in V$, v can be expressed as $v = f_1x_1 + f_2x_2 + \cdots + f_{n-1}x_{n-1} + f_nx_n$, where $f_i \in F$ and $x_i \in X$.*

Definition 2.3 (Basis). *A subset B is a basis of vector space V over field F if B is both linearly independent, and also spans across all of V.*

Definition 2.4 (Dimension)**.** *The dimension of a vector space (e.g.: reals, complex numbers, quaternions...) is the number of vectors in the basis of the vector space.*

Definition 2.5 (Commutative Property)**.** *The commutative property states that* $a \cdot b = b \cdot a$ *for all* $a, b \in A$ *for some set* A

Definition 2.6 (Associative Property)**.** *The associative property states that* $a \cdot (bc) = (ab) \cdot c$ *for all* $a, b, c \in A$ *for some set* A

Definition 2.7 (Alternativity)**.** *Alternativity is a property that* $x \cdot (xy) = (xx) \cdot y$ *and* $(yx) \cdot x = y \cdot (xx)$ *for all* $x, y \in A$ *given some set* A*.*

Note that alternativity is a weaker form of associativity.

Definition 2.8 (Power associativity)**.** *Power associativity is a property that states that the subset generated by any single element in a set is associative. For example,* $x \in X$*, if* x, x^2, x^3, x^4 *is the subset generated by the element* x*, then* $(xx)(xx) = x(xx)x = x(xxx) = (xxx)x$*.*

Definition 2.9 (Left Zero divisor)**.** *In a ring, an element* $a \in R$ *is a left zero divisor if there exists a nonzero* x *such that* $ax = 0$*.*

Definition 2.10 (Right Zero divisor)**.** *In a ring, an element* $a \in R$ *is a right zer odivisor if there exists a nonzero* y *such that* $ya = 0$*.*

If an element is either a left or a right zero divisor, it is simply referred to as a zero divisor, and if it is both a left and a right zero divisor, then it is referred to as a two-sided zero divisor.

Note that in a two-sided zero divisor, the multiplicands x and y need not be equal.

If the ring is commutative, and an element is a zero divisor, then it is necessarily a two-sided zero divisor, as $ax = xa = 0$.

3 The Cayley–Dickson construction

3.1 The Real Numbers

Consider the real numbers, of order 1. They are simply scalars, and can be expressed on a 1-Dimensional number line.

Proposition 3.1. *The Real numbers have associativity, commutativity, alternativity, and power associativity, but do not have nontrivial zero divisors.*

We will not prove these because these are evident by the way that we define multiplication over the real numbers.

3.2 Constructing The Complex Numbers

The first step of the Cayley-Dickson construction is to consider the complex numbers as a pair of real numbers.

Definition 3.1. *The set of complex numbers $\mathbb{C}$ is all elements of the form $a + bi$, where $a, b \in \mathbb{R}$, and $i = \sqrt{-1}$. Let $a + bi$ be equivalent to (a,b).*

The complex numbers satisfy the commutativity, associativity, power associativity, and alternativity. We will use that to discuss arithmetic over the complex numbers.

Proposition 3.2 (Addition over Complex Numbers)**.** $(a, b) + (c, d) = (a + c, b + d)$

This is because addition over the complex numbers simply works componentwise, so $(a + bi) + (c + di) = a + c + (b + d)i$.

Proposition 3.3 (Multiplication over Complex Numbers)**.** $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$

Proof. Because complex numbers satisfy the distributive property, we have: $(a+bi)(c+di) = ac+adi+bci+bdi^2$, and using the definition $i^2 = -1$, we can simplify this to $ac-bd+(ad+bc)i$, which is equivalent to $(ac - bd, ad + bc)$. □

Definition 3.2 (Conjugates for Complex Numbers)**.** *The conjugate of a complex number $x = a + bi \in \mathbb{C}$ is $x^* = a - bi$.*

Proposition 3.4. *The complex conjugate is defined so that $x \cdot x^* = n^2$, where n is the norm of the vector formed by the complex number.*

Proof. Given an arbitrary $x = a + bi$, $x \cdot x^* = (a + bi)(a - bi) = a^2 - b^2 i^2 = a^2 + b^2$, or $(a^2 + b^2, 0)$, which is a scalar. □

Because a complex number is defined by 2 real numbers (the scalar and the coefficient of i), it is a 2-Dimensional vector space.

3.3 Constructing the Quaternions

The next step of the Cayley-Dickson construction is to create the quaternions. Similarly to how complex numbers are constructed using two pairs of real numbers, the quaternions are constructed using two pairs of complex numbers, thus creating a fourth dimension.

Proposition 3.5 (Multiplication over Quaternions)**.** $(a, b) \cdot (c, d) = (ac - d^*b, da - bc^*)$

The order of the terms in the product is slightly different than it is for complex numbers. However, this is important for the conjugate to work.

Proposition 3.6 (Conjugates for Quaternions)**.** *The conjugates $(a, b)^*$ of (a, b) is $(a^*, -b)$.*

So if we multiply $(a, b)^*(a, b)$ we get:

$$\begin{aligned}(a, b)^*(a, b) &= (a^*, -b)(a, b)\\ &= (a^*a - b^*b, ba^* - ba^*)\\ &= (|a|^2 + |b^2|, 0)\end{aligned}$$

Thus we have a norm of some kind for quaternions, which we can in turn use to create an inverse. With all the pieces we need, we have created a new algebra that extends beyond the real and complex numbers. In the following section we will also describe multiplication and conjugates with more detail.

4 Properties of Quaternions

Definition 4.1. *Quaternions are numbers of the form $a + bi + cj + dk$ where i, j, and k satisfy the formula $i^2 = j^2 = k^2 = ijk = -1$.*

Similarly to how we describe complex numbers as $a + bi$, a quaternion is defined as $a+bi+cj+dk$ where a, b, c, and d are real numbers. This can be split into $bi+cj+dk$, the vector part, and a, the scalar part. The set of quaternions is a 4 dimensional vector space for each of the scalar, i, j, and k directions. The subset $\{1, i, j, k\}$ acts as the basis for the quaternions, because it spans across all of the quaternions and also is linearly independent $(a + bi + cj + dk = 0$ if and only if $a = b = c = d = 0$.

The act of adding two quaternions is physically equivalent to concatenating them, or connecting the beginning of one quaternion to the end of another to find a sum quaternion. Scalar multiplication is equivalent to physically stretching or shortening a quaternion while keeping its orientation/angle with respect to the planes the same.

Addition and scalar multiplication for quaternions is intuitive—it acts component wise:

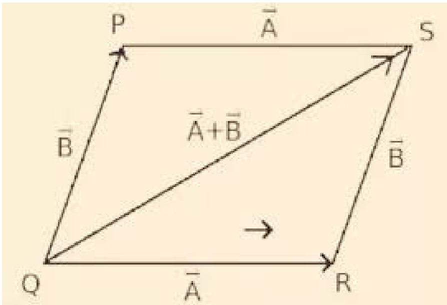
$$(a_1 + b_1i + c_1j + d_1k) + (a_2 + b_2i + c_2j + d_2k) = (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k$$

and

$$\lambda(a + bi + cj + dk) = \lambda a + (\lambda b)i + (\lambda c)j + (\lambda d)k.$$

The act of adding two quaternions is physically equivalent to concatenating them, or connecting the beginning of one quaternion to the end of another to find a sum quaternion. Scalar multiplication is equivalent to physically stretching or shortening a quaternion while keeping its orientation/angle with respect to the planes the same.

Although quaternions and vectors are inherently different, the addition of quaternions and vectors function in the same way. The concatenation of the vectors as described above act as shown. It is very important to note however, that $A + B \neq B + A$ for quaternions, as commutativity is not a property of quaternions. This is shown on the image below.



Multiplication of two quaternions is called the *Hamilton product* and uses the distributive property combined with the products of the elements of the basis. These are defined to be: $ij = k, jk = i, ki = j, ji = -k, kj - -i,$ and $ik = -j$. Therefore we have two quaternions $(a_1 + b_1 i + c_1 j + d_1 k)(a_2 + b_2 i + c_2 j + d_2 k) =:$

$$
\begin{aligned}
& a_1a_2 + a_1b_2i + a_1c_2j + a_1d_2k \\
+ \quad & b_1a_2i + b_1b_2i^2 + b_1c_2ij + b_1d_2ik \\
+ \quad & c_1a_2j + c_1b_2ji + c_1c_2j^2 + c_1d_2jk \\
+ \quad & d_1a_2k + d_1b_2ki + d_1c_2kj + d_1d_2k^2 \\
\\
= \quad & a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 \\
+ \quad & (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)i \\
+ \quad & (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)j \\
+ \quad & (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)k
\end{aligned}
$$

The multiplication of two quaternions is noncommutative as shown. This is because the elements of the basis are no commutative themselves, which affects the entire quaternion number system.

Thus, when multiplying $(a_1 + a_2 i + a_3 j + a_4 k)(b_1 + b_2 i + b_3 j + b_4 k)$, we get: $n = a \times b = n_1 + n_2 + n_3 + n_4$, where:

$$
\begin{aligned}
n_1 &= b_1a_1 - b_2a_2 - b_3a_3 - b_4a_4 \\
n_2 &= b_1a_2 + b_2a_1 - b_3a_4 - b_4a_3, \\
n_3 &= b_1a_3 + b_2a_4 + b_3a_1 - b_4a_2, \text{and} \\
n_4 &= b_1a_4 - b_2a_3 + b_3a_2 + b_4a_1.
\end{aligned}
$$

Definition 4.2. *The norm of a quaternion* $a_1 + a_2 i + a_3 j + a_4$ *is* $\|a\| = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}$.

Norms represent the distance a quaternion is from the origin. The norm is important in finding the conjugate.

Proposition 4.1. *The conjugate of* $a_1 + a_2 i + a_3 j + a_4$ *is* $a_1 - a_2 i - a_3 j - a_4 k$.

Proof. $\bar{a} \cdot a = a_1^2 + a_2^2 + a_3^2 + a_4^2$. Thus, given the general formula for Hamiltonian multiplication given above, we have a system of equations for the coefficients n_1, n_2, n_3, and n_4. Our system of equations is $n_1 =$, $n_2 = 0$, $n_3 = 0$, and $n_4 = 0$. Because the calculations are tedious, we will not go through them at length (it's simply solving a system of equations of 4 variables with 4 equations, where we get that , $\bar{a}_2$, $\bar{a}_3$, and $\bar{a}_4$ are $a_1, -a_2, -a_3$, and $-a_4$ respectively. □

Conjugates are useful because multiplying a quaternion by its conjugate yields a real number.

However, rearranging our result above yields a nice result. Unlike for complex numbers, the conjugate of a quaternion q^* can be represented in terms of q:

$$q^* = -\frac{1}{2}(q + iqi + jqj + kqk)$$

Given the conjugate of quaternion, finding the quaternion's inverse is very easy. Because $q \cdot q^* = a_1^2 + a_2^2 + a_3^2 + a_4^2$. This means that $q \cdot \frac{q^*}{a_1^2+a_2^2+a_3^2+a_4^2} = 1$, and thus that:

Definition 4.3. *The inverse quaternion q^{-1} of q is defined in terms of its conjugate as:*

$$q^{-1} = \frac{q^*}{\|q\|^2}.$$

Every quaternion can be written as a product of its *unit quaternion* $\mathbf{U}_q = \frac{q}{\|q\|}$ and its norm. In other words, $q = \|q\| \cdot \mathbf{U}_q$. Then:

$$\begin{aligned}
q^{-1}q &= \frac{q^*}{\|q\|^2} \cdot \|q\| \cdot \mathbf{U}_q \\
&= \frac{q^*}{\|q\|} \cdot \mathbf{U}_q \\
&= \frac{q^* q}{\|q\|^2} \\
&= 1.
\end{aligned}$$

5 Rotations

5.1 Rotations in 2D space

Definition 5.1 (Euler's Formula)**.** $e^{i\pi} = cos(\theta) + isin(\theta)$

Proof. Using the power series for e^x,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + ...$$

we have that

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \frac{(ix)^8}{8!} + ...$$

which evaluates to

$$1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} = \frac{ix^7}{7!} + \frac{x^8}{8!}$$

grouping this, we get

$$(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots)$$

which yields $cos(\theta) + isin(\theta)$ □

Euler's formula can be used for rotations in the 2-dimensional plane. Because the unit circle can be represented as $e^{i\theta}$, and because of the laws of exponents, multiplying two exponents will result in a rotation.

Example. Multiplying $e^{i\theta}$ and $e^{i\alpha}$ yields $e^{i(\theta+\alpha)}$. For example, given $\theta = \pi/3$ and $\alpha = \pi/6$, we have $e^{i\pi/3)} \cdot e^{i\pi/6} = e^{i\pi/2}$. Essentially, the multiplication of these two exponents is like a group action, where $e^{i\pi/6}$ rotated the vector $(1/2, \sqrt{3}/2)$ by $\pi/6$ radians, to produce a vector $(0, 1)$. $\square$

5.2 Rotations in 3D space

Before we prove the quaternion rotation identity, we must define the cross and dot products for quaternions. In fact, the cross and dot products were originally created *for* quaternions before they became widely used in linear algebra.

To start, already know the component-wise multiplication of quaternions:

$$\begin{aligned}(a, b, c, d) \cdot (e, f, g, h) = (&ae - bf - cg - dh,\\ &be + af - dg + ch,\\ &ce + df + ag - bh,\\ &de - cf + bg + ah)\end{aligned}$$

where writing (a, b, c, d) is the same as $a + bi + cj + dk$. We will also denote the vector part of (a, b, c, d) as $\vec{x}$ and the vector part of (e, f, g, h) as $\vec{y}$. First, note that the scalar component of this quaternions product is $ae - (bf + cg + dh) = ae - \vec{x} \cdot \vec{y}$. Next, we can combine the terms be, ce, de, and af, ag, ah to get $e\vec{x} + a\vec{y}$. Finally, the remaining terms, $-dg + ch + df - bh - cf + bg$, are just the cross product $\vec{x} \times \vec{y}$. Putting this all together, we have:

$$(a, \vec{x}) \cdot (e, \vec{y}) = (ae - \vec{x} \cdot \vec{y}, e\vec{x} + a\vec{y} + \vec{x} \times \vec{y}). \tag{1}$$

Therefore, when the scalar parts are 0, we just have:

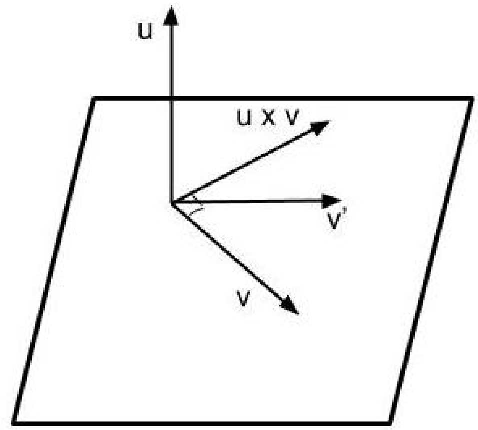
$$\vec{x} \cdot \vec{y} = (-\vec{x} \cdot \vec{y}, \vec{x} \times \vec{y}). \tag{2}$$

Now we are ready to approach quaternions as rotations in space.

Just as complex numbers can be used to create 2-dimensional rotations, quaternions are used for 3D rotation. Let $\vec{u}$ be the unit vector we wish to rotate the vector $\vec{v}$ around. Note that $\vec{u}$ and $\vec{v}$ are only the vector parts of a quaternion.

Lemma 5.1. *If $\vec{v}$ is orthogonal to $\vec{u}$, the equation $\vec{v'} = \cos(\alpha)\vec{v} + (\sin\alpha)(\vec{u} \times \vec{v})$ rotates $\vec{v}$ around $\vec{u}$ by the angle α.*

Proof. Reference the image on the right. If $\vec{v}$ is orthogonal to $\vec{u}$, we can imagine $\vec{v}$ is on the place defined by the normal vector $\vec{u}$. Therefore $\vec{u} \times \vec{v}$ and $\vec{v'}$ are also on this plane. In addition, $\|\vec{v}\| = \|\vec{v'}\| = \|\vec{u} \times \vec{v}\|$. That means $\vec{v}$, $\vec{v'}$, and $\vec{u} \times \vec{v}$ are all on a circle. Now to find the location of $\vec{v'}$, we project $\vec{v'}$ onto $\vec{v}$ and $\vec{u} \times \vec{v}$. These values are $\cos(\alpha)\vec{v}$ and $(\sin\alpha)(\vec{u} \times \vec{v})$ respectively. Thus $\vec{v'} = \cos(\alpha)\vec{v} + (\sin\alpha)(\vec{u} \times \vec{v})$.



So far we haven't used quaternions, so let's connect the formula we just proved into a simpler quaternion form.

Lemma 5.2. *Let the scalar parts be* 0 *and the vector parts of* v, v', *and* u *be* $\vec{v}$, $\vec{v'}$, *and* $\vec{u}$. *If* $\vec{v}$ *is orthogonal to* $\vec{u}$, *then the rotation formula becomes* $v' = e^{\alpha u} v$.

Proof. Recall from above that the quaternion $uv = (-u \cdot v, u \times v)$. Again since u and v are perpendicular, the scalar part of their product becomes 0. Thus $uv = \vec{u} \times \vec{v}$ and this can be substitutes into our result from Lemma 5.1:

$$\begin{aligned} \vec{v'} &= \cos(\alpha)\vec{v} + \sin(\alpha)(\vec{u} \times \vec{v}) \\ &= \cos(\alpha)\vec{v} + \sin(\alpha)uv \\ &= \big(\cos(\alpha) + \sin(\alpha)u\big)v \end{aligned}$$

This looks suspiciously Euler Formula-like. In fact, u has a peculiar property when we square it:

$$\begin{aligned} u^2 = (0, \vec{u})(0, \vec{u}) &= (-\vec{u} \cdot \vec{u}, \vec{u} \times \vec{u}) \\ &= (-\|u\|^2, \vec{0}) \\ &= (-1, \vec{0}) \\ &= -1. \end{aligned}$$

Funnily enough, $u^2 = -1$ just like i! Because u has the same property as i, we can now complete the final step in our quaternion formula for $\vec{v'}$:

$$\begin{aligned} \vec{v'} &= \big(\cos(\alpha) + \sin(\alpha)u\big)v \\ &= e^{\alpha u} v. \end{aligned}$$

□

Theorem 5.3 (Rodrigues' rotation formula)**.** *The equation*

$$\vec{v'} = (1 - \cos\alpha)(\vec{v} \cdot \vec{u})\vec{u} + \cos(\alpha)\vec{v} + \sin(\alpha)(\vec{u} \times \vec{v})$$

rotates $\vec{v}$ *around* $\vec{u}$ *by the angle* α.

Proof. The key to this proof is to split up $\vec{v'}$ into $\vec{v'_\perp}$ and $\vec{v'_\parallel}$, which represent the parts of $\vec{v'}$ that are perpendicular and parallel to $\vec{u}$. We can evaluate their locations separately. First, it is obvious that $\vec{v'_\parallel} = \vec{v}_\parallel$. Now to find $\vec{v'_\perp}$, we simply plug in our formula from Lemma 5.1. Therefore, so far we have:

$$\begin{aligned}\vec{v'} &= \vec{v_\parallel} + \cos(\alpha)\vec{v_\perp} + (\sin\alpha)(\vec{u} \times \vec{v_\perp}) \\ &= \vec{v_\parallel} + \cos(\alpha)(\vec{v} - \vec{v_\parallel}) + (\sin\alpha)(\vec{u} \times \vec{v}).\end{aligned}$$

Here, the substitution of $\vec{v}$ for $\vec{v_\perp}$ in the cross product can be made since $\vec{u} \times \vec{v} = \vec{u} \times \vec{v_\perp} + \vec{u} \times \vec{v_\parallel} = \vec{u} \times \vec{v_\perp}$. We continue to try to eliminate any more $\vec{v_\perp}$s or $\vec{v_\parallel}$s as follows:

$$\begin{aligned}\vec{v'} &= \vec{v_\parallel} + \cos(\alpha)(\vec{v} - \vec{v_\parallel}) + (\sin\alpha)(\vec{u} \times \vec{v}) \\ &= (1 - \cos\alpha)(\vec{v} \cdot \vec{u})\vec{u} + \cos(\alpha)\vec{v} + \sin(\alpha)(\vec{u} \times \vec{v}).\end{aligned}$$

Thus we have our result. □

Clearly, this formula is not very nice. That's why we also have the much nicer *real* quaternion rotation formula that makes use of Lemma 5.2.

Theorem 5.4. *The quaternion formula $v' = e^{\frac{\alpha}{2}u} v e^{-\frac{\alpha}{2}u}$ represents v rotated around u by the angle α.*

Proof. We will backtrack a bit from Rodrigues' rotation formula and start with $v' = v_\parallel + v'_\perp$. Then:

$$\begin{aligned}v' &= v_\parallel + e^{\alpha u} v_\perp \\ &= e^{\frac{\alpha}{2}u} e^{-\frac{\alpha}{2}u} v_\parallel + e^{\frac{\alpha}{2}u} e^{\frac{\alpha}{2}u} v_\perp \\ &= e^{\frac{\alpha}{2}u} v_\parallel e^{-\frac{\alpha}{2}u} + e^{\frac{\alpha}{2}u} v_\perp e^{-\frac{\alpha}{2}u}.\end{aligned}$$

In this last step, we were able to do two things: first, we used the fact that $e^{\frac{\alpha}{2}u} v_\perp = v_\perp e^{-\frac{\alpha}{2}u}$. This can be proved by simple calculation. Second, it is also true that $v_\parallel e^{\frac{\alpha}{2}u} = e^{\frac{\alpha}{2}u} v_\parallel$, which is usually not true for the quaternions. In this scenario it is, however, because it turns out that for every two quaternions A and B, we have $AB - BA = 2(\vec{A} \times \vec{B})$. Here, the vector part of $e^{\frac{\alpha}{2}}$ is $\sin(\alpha)u$, which is parallel to $v_\parallel$.

Now we continue to the remainder of the proof:

$$\begin{aligned}v' &= e^{\frac{\alpha}{2}u} v_\parallel e^{-\frac{\alpha}{2}u} + e^{\frac{\alpha}{2}u} v_\perp e^{-\frac{\alpha}{2}u} \\ &= e^{\frac{\alpha}{2}u} (v_\parallel + v_\perp) e^{-\frac{\alpha}{2}u} \\ &= e^{\frac{\alpha}{2}u} v e^{-\frac{\alpha}{2}u}.\end{aligned}$$

□