PROOF THAT SOME PROJECTIVE SPECIAL LINEAR GROUPS ARE SIMPLE

ALEX THOLEN

1. What are PSL's?

Projective special linear groups are one family of infinite simple groups. What are projective special linear groups, you may ask?

Definition 1.1. The *special linear group* $\mathrm{SL}_n(F)$ is the set of $n \times n$ matrices with determinant 1 of the field F.

Definition 1.2. The *projective special linear group* $\mathrm{PSL}_n(F)$ is the quotient group of $\mathrm{SL}_n(F)$ by its center: $\mathrm{PSL}_n(f) = \mathrm{SL}_n(F)/Z(\mathrm{SL}_n(F))$

In case you have forgotten, the center of a group is defined as follows:

Definition 1.3. The center of a group G is the is the set of elements $Z(G)$ that commute with every element of G, i.e.

$$Z(G) = \{h \in G : gh = hg \text{ for all } g \in G\}.$$

2. Necessary Steps

Now, to prove that most projective special linear groups are simple, we will use double transitivity.

Definition 2.1. An action of a group G on a set X is called *transitive* when it has the property where for any $x, y \in X$ there exists a $g \in G$ such that $g(x) = y$. It is called *doubly transitive* when for any $x_1, x_2, y_1, y_2 \in X$ with distinct x's and y's there exists a $g \in G$ such that $g(x_1) = y_1, g(x_2) = y_2$.

Theorem 2.2. *If G acts doubly transitively on X then the stabilizer subgroup of any point in X is a maximal subgroup of G.*

Proof. Pick $x \in X$ and let $H_x = \mathrm{Stab}_x$. The first step is to show that for any $g \notin H_x$, $G = H_x \cup H_x g H_x$. We have $g' \in G$ and $g' \notin H_x$, and the goal is to show that $g' \in H_x g H_x$. We know that gx and $g'x$ aren't x, and so via double transitivity we get that $g''x = x$ and $g''(gx) = g'x$ for some $g'' \in G$. We know that $g'' \in H_x$ and so ■

Theorem 2.3. *Suppose G acts doubly transitively on a set X. Any normal subgroup $N \triangleright G$ acts on X either trivially or transitively.*

Proof. Suppose N does not act trivially: $nx \neq x$ for some $x \in X$ and some $n \neq 1$ in N. Pick any y and y' in X with $y \neq y'$. By double transitivity, there is $g \in G$ such that $gx = y$ and $g(ny) = y'$. Then $y' = (gng^{-1})(gx) = (gng^{-1})y$ and $gng^{-1} \in N$, so N acts transitively on X. ■

Date: June 14, 2020.

Theorem 2.4. *(Iwasawa). Let G act doubly transitively on a set X. Assume the following:*

(1) For some $x \in X$ the group $Stab_x$ has an abelian normal subgroup whose conjugate subgroups generate G.

(2) $[G,G] = G$.

Then G/K is a simple group, where K is the kernel of the action of G on X.

Proof. To show G/K is simple we will show the only normal subgroups of G lying between K and G are K and G. Let $K \subset N \subset G$ with $N \triangleright G$. Let $H = \mathrm{Stab}_x$, so H is a maximal subgroup of G. Sinec N is normal, $NH = \{nh : n \in N, h \in H\}$ is a subgroup of G, and it contains H, so by maximality either $NH = H$ or $NH = G$. And so N acts trivially or transitively on X. If $NH = H$ then $N \subset H$, so N firxes x. Therefore N does not act transitively on X so N must act trivially on X, which implies $N \subset K$. Since $K \subset N$ by hypothesis, we have $N = K$. Now suppose $NH = G$. Let U be the abelian normal subgroup of H in the hypothesis: its conjugate subgroups generate G. Since $U \triangleright H, NU \triangleright NH = G$. Then for $g \in G, gUg^{-1} \subset g(NU)g^{-1} = NU$, which shows that NU contains all the conjugate subgroups of U. By hypothesis it follows that $NU = G$. Thus $G/N = (NU)/N \approx U/(N \cap U)$. Since U is abelian, the isomorphism tells us that G/N is abelian, so $[G,G] \subset N$. Since $G = [G,G]$ by hypothesis, we have $N = G$. ∎

3. Is $\mathrm{PSL}_2(F)$*simple*?

Theorem 3.1. *The action of $\mathrm{SL}_2(F)$ on the linear subspaces of F^2 is doubly transitive.*

Proof. An obvious pair of distinct linear subspaces in F^2 is $F\binom{1}{0}$ and $F\binom{0}{1}$. It suffices to show that, given any two distinct linear subspaces F_v and F_w, there is an $A \in \mathrm{SL}_2(F)$ that sends $F\binom{1}{0}$ to F_v and $F\binom{0}{1}$ to F_w, because we can then use $F\binom{1}{0}$ and $F\binom{0}{1}$ as an intermediate step to send any pair of distinct linear subspaces to any other pair of distinct linear subspaces. Let $v = \binom{a}{c}$ and $w = \binom{b}{d}$. Since $F_v \neq F_w$, the vectors v and w are linearly independent, so $D := adbc$ is nonzero. Let $A = \binom{ab/D}{cd/D}$, which has determinant $a(d/D)(b/D)c = D/D = 1$, so $A \in \mathrm{SL}_2(F)$. Since $A\binom{1}{0} = \binom{a}{c} = v$ and $A\binom{0}{1} = \binom{b/D}{d/D} = (1/D)w$, A sends $F\binom{1}{0}$ to Fv and $F\binom{0}{1}$ to $F(1/D)w = Fw$. ∎

We will apply Iwasawa's criterion (**??**wasawa) to $\mathrm{SL}_2(F)$ acting on the set of linear subspaces of F^2. This action is doubly transitive by the previous theorem. It remains to check the following:

- the kernel K of this group is the center of $\mathrm{SL}_2(F)$, so $\mathrm{SL}_2(F)/K = \mathrm{PSL}_2(F)$,
- the stabilizer subgroup of $\binom{1}{0}$ contains an abelian normal subgroup whose conjugate subgroups generate $\mathrm{SL}_2(F)$,$[\mathrm{SL}_2(F), \mathrm{SL}_2(F)] = \mathrm{SL}_2(F)$.

Theorem 3.2. *The kernel of the action of $\mathrm{SL}_2(F)$ on the linear subspaces of F^2 is the center of $\mathrm{SL}_2(F)$.*

- *Proof.* A matrix $\begin{pmatrix} ab \\ cd \end{pmatrix} \in \mathrm{SL}_2(F)$ is in the kernel K of the action when it sends each linear

subspace of F^2 back to itself. If the matrix preserves the lines $F\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $F\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then $c = 0$ and $b = 0$, so $\begin{pmatrix} ab \\ cd \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$. The determinant is 1, so $d = 1/a$. If $\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$

preserves the line $F\begin{pmatrix}1\\1\end{pmatrix}$ then $a = 1/a$, so $a = \pm 1$. This means $\begin{pmatrix}ab\\cd\end{pmatrix} = \pm\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$. Conversely, the matrices $\pm\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ both act trivially on the linear subspaces of F^2, so $K = \left\{\pm\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right\}$ If a matrix $\begin{pmatrix}ab\\cd\end{pmatrix}$ is in the center of $\mathrm{SL}_2(F)$ then it commutes with $\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}$, which implies $a = d$ and $b = c$ (check!). Therefore $\begin{pmatrix}ab\\cd\end{pmatrix} = \begin{pmatrix}a\\0\\a\end{pmatrix}$. since this has determinant 1 $a^2 = 1$, so $a = \pm 1$. Conversely, $\pm\begin{pmatrix}1\\0\\1\end{pmatrix}$ commutes with all matrices. Let $x = F\begin{pmatrix}1\\0\end{pmatrix}$. Its stabilizer subgroup in $\mathrm{SL}_2(F)$ is

$$\begin{aligned}\mathrm{Stab}_{F(1)} &= \left\{A \in \mathrm{SL}_2(F) : A\begin{pmatrix}1\\0\end{pmatrix} \in F\begin{pmatrix}1\\0\end{pmatrix}\right\}\\ &= \left\{\begin{pmatrix}a & b\\0 & d\end{pmatrix} \in \mathrm{SL}_2(F)\right\}\\ &= \left\{\begin{pmatrix}a & b\\0 & 1/a\end{pmatrix} : a \in F^\times, b \in F\right\}\end{aligned}$$

This subgroup has a normal subgroup

$$U = \left\{\begin{pmatrix}1 & *\\0 & 1\end{pmatrix}\right\} = \left\{\begin{pmatrix}1 & \lambda\\0 & 1\end{pmatrix} : \lambda \in F\right\}$$

which is abelian since $\begin{pmatrix}1 & \lambda\\0 & 1\end{pmatrix}\begin{pmatrix}1 & \mu\\0 & 1\end{pmatrix} = \left\{\begin{pmatrix}1 & \lambda\\0 & 1\end{pmatrix} : \lambda \in F\right\}$.

■

Theorem 3.3. *The subgroup U and its conjugates generate $\mathrm{SL}_2(F)$. More precisely, any matrix of the form $\begin{pmatrix}1 & 0\\ & 1\end{pmatrix}$ is conjugate to a matrix of the form $\begin{pmatrix}1 & *\\0 & 1\end{pmatrix}$, and every element of $\mathrm{SL}_2(F)$ is the product of at most 4 elements of the form $\begin{pmatrix}1 & 0\\ & 1\end{pmatrix}$ or $\begin{pmatrix}1 & *\\0 & 1\end{pmatrix}$.*

Proof. The matrix $\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$ is in $\mathrm{SL}_2(F)$ and $\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & \lambda\\0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}^{-1} = \begin{pmatrix}1 & 0\\-\lambda & 1\end{pmatrix}$, so $\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$ conjugates $U = \{\begin{pmatrix}1 & *\\0 & 1\end{pmatrix}\}$ to the group of lower triangular matrices $\{\begin{pmatrix}1 & 0\\ & 1\end{pmatrix}\}$. Pick $\begin{pmatrix}a & b\\c & d\end{pmatrix}$ in $\mathrm{SL}_2(F)$. To show that it is a product of matrices of type $\begin{pmatrix}1 & 0\\ & 1\end{pmatrix}$ or $\begin{pmatrix}1 & *\\0 & 1\end{pmatrix}$, first suppose $b \neq 0$. Then

$$\begin{pmatrix}a & b\\c & d\end{pmatrix} = \begin{pmatrix}1 & 0\\(d-1)/b & 1\end{pmatrix}\begin{pmatrix}1 & b\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\(a-1)/b & 1\end{pmatrix}.$$

If $c \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & (a-1)/c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} 1 & (d-1)/c \\ 0 & 1 \end{pmatrix}.$$

If $b = c = 0$ then the matrix is $\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$, and

$$\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (1-a)/a & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a-1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1/a \\ 0 & 1 \end{pmatrix}$$

■

So far F has been any field. Now we specify $\#F \geq 4$.

Theorem 3.4. *If* $\#F \geq 4$ *then* $[\mathrm{SL}_2(F), \mathrm{SL}_2(F)] = \mathrm{SL}_2(F)$

Proof. We compute an explicit commutator: $\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}^{-1} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & b\,(a^2-1) \\ 0 & 1 \end{pmatrix}$ Since $\#F \geq 4$, there is an $a \neq 0, 1$, or -1 in F, so $a^2 \neq 1$. Using this value of a and letting b run over F shows $[\mathrm{SL}_2(F), \mathrm{SL}_2(F)]$ contains U. Since the commutator subgroup is normal, it contains every subgroup conjugate to U, so $[\mathrm{SL}_2(F), \mathrm{SL}_2(F)] = \mathrm{SL}_2(F)$ by 3.3. ■

3.4 is false when $\#F = 2$ or $3 : \mathrm{SL}_2(\mathrm{F}_2) = \mathrm{GL}_2(\mathrm{F}_2)$ is isomorphic to S_3 and $S_3, S_3] = A_3$. In $\mathrm{SL}_2(\mathrm{F}_3)$ there is a unique 2 -Sylow subgroup, so it is normal, and its index is 3, so the quotient by it is abelian. Therefore the commutator subgroup of $\mathrm{SL}_2(\mathrm{F}_3)$ lies inside the 2 -Sylow subgroup (in fact, the commutator subgroup is the 2 -Sylow subgroup).

Theorem 3.5. *If* $F \geq 4$ *then the group* $\mathrm{PSL}_2(F)$ *is simple.*

Proof. By the previous four theorems the action of $\mathrm{SL}_2(F)$ on the linear subspaces of F^2 satisfies the hypotheses of Iwasawa's theorem, and its kernel is the center of $\mathrm{SL}_2(F)$ ■

This can be continued on to show for $SL_n(F)$ where $n > 2$, but that is beyond the scope of this.

References:

Wikipedia - to obtain very basic information of what was needed (and to look at the monster group)

https://kconrad.math.uconn.edu/blurbs/grouptheory/PSLnsimple.pdf - main source.