Ring Theory

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1 Rings

1.1 Definition of a Ring

A ring is a set that has the two operations multiplication (x) and addition (+). It is also a field that doesn't have to be commutative or have an identity for each element.
It satisfies these properties:

1. Associativity for multiplication : $(a \times b) \times c = a \times (b \times c)$
2. (R,+) is an abelian group.
3. $(a + b) \times c = (a \times c) + (b \times c)$ and $a \times (b + c) = (a \times b) + (a \times c)$.

1.2 Types of Rings

Let R be a ring.
A ring :

1. Is commutative if multiplication is commutative.
2. has an identity (1) if $1 \in R$ st for all $a \in R$ 1 × a = a × 1.
3. is a division ring if it has an identity 1 st $1 \neq 0$ and for all $a \in R$ there exists a $b \in R$ such that a× b= b ×a = 1.
4. Is a field if it is both commutative and a division ring.

1.3 Examples of Rings

Some examples of rings are

1. Trivial rings are when you take R to be a commutative group with addition as the operation and defining multiplication for all $a, b \in R$ a × b = 0. If the original group was R = 0, then it's a special type of trivial ring called the zero ring.
2. The integers $\mathbb{Z}$ are a commutative ring with identity 1, but they are not a division ring.
3. The even integers $2\mathbb{Z}$ is an example of a ring. However this ring doesn't contain an identity.

4. The integers mod n.

5. The rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$, and complex numbers $\mathbb{C}$ are all commutative and division rings, which means they are fields.

1.4 Properties and elements of Rings

1. The zero divisor of a ring is an element, that's not zero, such that $a \times b = b \times a = 0$.

2. If R has identity $1 \neq 0$. An element is a unit in R if $\exists$ some U st UV =VU =1.

3. An integral domain is a commutative ring with identity $1 \neq 0$ and has no zero divisors.

4. A subring is a subgroup closed under multiplication.

5. Let I be a subset of the ring R.

    - I is a a left ideal if I is a subring of R, closed under left multiplication.
    - I is a right ideal if I is a subring of R, closed under right multiplication.
    - I is an ideal if I is a subring of R closed under both left and right multiplication.

1.5 Quadratic Integer Rings

If D is a square free integer, which means it isn't divisible by any perfect squares excluding one, then the subset $\mathbb{Z}[D] = [a + b\sqrt{D}]$ is a subring of the $\mathbb{Q}[D]$, also known as the quadratic field. $\mathbb{Z}[D] = [a + b\frac{\sqrt{D}+1}{2}]$ is also a subring if $D \equiv 1 mod 4$. From this you can get something called the ring of integers where $\mathbb{Z}[D] = [a + b\sqrt{D}]$ if $D \equiv 2, 3 mod 4$ and $\mathbb{Z}[D] = [a + b\frac{\sqrt{D}+1}{2}]$ if $D \equiv 1 mod 4$.