Further Topics in Finite Group Theory

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1 Nilpotent Groups

Given a group G and $H \leq G$, the **commutator subgroup** of G and H, denoted $[G, H]$ is the subgroup generated by all elements of the form $ghg^{-1}h^{-1}$ where $g \in G$ and $h \in H$, that is

$$[G, H] = \langle ghg^{-1}h^{-1} : g \in G, h \in H \rangle$$

When a second group is unspecified, the commutator subgroup of G, also called the **derived subgroup** of G, is defined to be $[G, G]$. $[G, G] \trianglelefteq G$ and $G/[G, G]$ is abelian, but the proofs of these are simple and not very interesting so they will be omitted. A **lower central series** for a group G is defined as a series with

$$G = G_0 \geq G_1 \geq G_2 ...$$

where $G_{i+1} = [G_i, G]$. There are similarly defined **upper central series** and **central series** which exist if and only if a lower central series exists, but they will not be discussed here. A group G is called **nilpotent** if it has a lower central series so that $G_n = \{e\}$ for some finite n, where e is the identity.

Theorem 1.1: If G is nilpotent and $H \leq G$, $N_G(H) \neq H$ unless $H = G$, where $N_G(H)$ denotes the normalizer of H in G.

Proof: Since G is nilpotent, it has a lower central series so that

$$G = G_0 \geq G_1 \geq G_2 \geq ...G_n = \{e\}$$

There exists some G_i where $G_i \leq H$ but $G_{i-1} \nleq H$. Now

$$[G_{i-1}, H] \leq [G_{i-1}, G] = G_i \leq H$$

so $ghg^{-1}h^{-1} \in H$ for all $g \in G_{i-1}$ and $h \in H$. Since H is a subgroup, $ghg^{-1} \in H$ for all $h \in H$, that is $gHg^{-1} = H$ for all $g \in G_{i-1}$. It follows from the definition of the normalizer that $G_{i-1} \leq N_G(H)$. Since $H \nleq G_{i-1}$, there must be some element of G_{i-1} not in H so there must be some element in $N_G(H)$ not in H, hence $H \neq N_G(H)$. ■

The converse of the above theorem is true for finite groups, but the proof is more complicated so it is omitted. In addition to Theorem 1.1, nilpotent groups satisfy several other useful properties. It can be proven via central series that nilpotent groups are all solvable. Also, the Sylow subgroups of a nilpotent group satisfy some interesting properties.

Theorem 1.2: Let G be a nilpotent group and P a Sylow p-subgroup. $P \trianglelefteq G$.

Proof: Since P is a Sylow p-subgroup, the second Sylow theorem states that all other Sylow p-subgroups are conjugate to P. Thus, if $g \notin N_G(P)$, gPg^{-1} is some other Sylow p-subgroup. Since $P \trianglelefteq N_G(P)$, it is the only Sylow p-subgroup in $N_G(P)$ so $gPg^{-1} \nleq N_G(P)$ and similarly $gN_G(P)g^{-1} \neq N_G(P)$. Therefore, $N_G(P)$ is fixed by conjugation only by its own elements, i.e. $N_G(N_G(P)) = N_G(P)$. Since G is nilpotent, we $N_G(P) = G$ by Theorem 1.1 and hence $P \trianglelefteq G$. ■

Again, for finite groups the converse is true. This can be proved most easily using upper central series, but as upper central series were omitted so will the proof of the converse.

Theorem 1.3: Let G be a finite group. Every Sylow subgroup of G is normal if and only if G is isomorphic to the direct product of its Sylow subgroups.

Proof: First, assume every Sylow subgroup of G is normal. Therefore, for a given prime p, there is only one Sylow p-subgroup. If P_1 and P_2 are distinct Sylow subgroups, $P_1 \cap P_2 = \{e\}$. Thus $|P_1P_2| = |P_1||P_2|$ and similarly

$$|P_1P_2...P_n| = |P_1||P_2|...|P_n| = |G|$$

where the product ranges over all of the Sylow subgroups of G. Since each element of $P_1P_2...P_n$ is an element of G, it is a subset of G. The only subset of G with $|G|$ elements is G itself, so

$$P_1P_2...P_n = G$$

Since each P_i is normal and have pairwise trivial intersections,

$$G \cong P_1P_2...P_n \cong P_1 \times P_2 \times ...P_n$$

The converse follows from the fact that every factor in a direct product is normal.■

Corollary 1.4: If G is nilpotent, G is isomorphic to the direct product of its Sylow subgroups.

Proof: This follows immediately from Theorems 1.2 and 1.3.■

Corollary 1.5: If G is nilpotent, any elements of coprime order commute.

Proof: From Corollary 1.4, G can be written as a direct product of its Sylow subgroups. Let $a = (a_1, a_2, ...a_n)$ and $b = (b_1, b_2, ...b_n)$ where each term in the ordered n-tuple represents a Sylow subgroup of G. If a and b have coprime order, given some i at least one of a_i or b_i is the identity. Since the identity commutes with every element, it follows from the definition of multiplying the ordered n-tuples that $ab = ba$. ■

2 Maximal Subgroups

The definition of a maximal subgroup is fairly intuitive. $M < G$ (note that $M \neq G$) is called a **maximal subgroup** if there is no subgroup H where $M < H < G$. For a nilpotent group G, if M is maximal $M < N_G(M) \leq G$ so $N_G(M) = G$ and hence $M \trianglelefteq G$. If G is not nilpotent, this result may not be true (for example, take any maximal subgroup of A_5). This leads to the definition of the Frattini subgroup. The **Frattini subgroup** of a group G, denoted $\Phi(G)$, is the intersection of all of its maximal subgroups, that is to say

$$\Phi(G) = \bigcap_M M$$

where the intersection is over each maximal subgroup M. Unlike maximal subgroups, the Frattini subgroup *is* normal, and in fact a stronger result is true.

Theorem 2.1: $\Phi(G)$ is fixed under every automorphism of G; in particular, it is fixed under conjugation and thus is normal.

Proof: Let $f : G \to G$ be an automorphism. Now if M is maximal, $f(M)$ must also be maximal, since

$$M < H < G \iff f(M) < f(H) < f(G) = G$$

If $A \neq B$, $f(A) \neq f(B)$ since automorphisms are bijections. Therefore, f maps each distinct maximal subgroup to another distinct maximal subgroup. Also, if $a \in U$ and $a \in V$, $f(a) \in f(U)$ and $f(a) \in f(V)$ so $f(U \cap V) = f(U) \cap f(V)$, so

$$f(\bigcap_M M) = \bigcap_M f(M)$$

where the intersection is over each maximal subgroup M. Since f just permutes the maximal subgroups, rearranging the intersection yields

$$\bigcap_M M = \Phi(G) \qquad \blacksquare$$

3 Resources

1. Notes from the Euler Circle
2. `http://www-groups.mcs.st-and.ac.uk/~colva/topics/ch7.pdf`
3. `http://www1.spms.ntu.edu.sg/~frederique/chap1.pdf`