Pólya Enumeration Theorem

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1 Burnside's Lemma

Let G be a finite group that acts on a set X. Then,

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|\text{fix}(g)|$$

where fix(g) is the is the set of elements in X that are fixed by g and $|X/G|$ denotes the number of orbits. In other words, the number of orbits is equal to the average number of elements that are fixed by an element of G.

1.1 Proof

Proof. $\sum_{g\in G}|\text{fix}(g)|$ enumerates the pairs $(g,x)\in G\times X$ where $g(x)=x$. Therefore,

$$\sum_{g\in G}|\text{fix}(g)| = |\{(g,x)\in G\times X|g(x)=x\}|$$
$$= \sum_{x\in X}|\text{stab}(x)| = \sum_{H\in X/G}\sum_{x\in H}|\text{stab}(x)|$$

Without loss of generality, suppose G acts on X from the left. If y, z are two elements from the same orbit, and α is an element of G such that $\alpha(y)=z$, then $g\mapsto \alpha g\alpha^{-1}$ is a bijection from stab(y) to stab(z). From the orbit-stabilizer theorem we get that for any i in the orbit H of X

$$\sum_{x\in H}|\text{stab}(x)| = \sum_{x\in H}|\text{stab}(i)| = |\text{orb}(i)|\cdot|\text{stab}(i)| = |G|$$

Therefore, we get that

$$\sum_{g\in G}|\text{fix}(g)| = \sum_{H\in X/G}\sum_{x\in H}|\text{stab}(x)| = \sum_{H\in X/G}|G| = |G|\cdot|X/G|$$

and

$$\frac{1}{|G|}\cdot|G|\cdot|X/G| = |X/G|$$

□

1.2 Example

Suppose we want to count the number rotationally distinct colorings of the faces of a cube there are using three colors. Let X be the ways we can color a cube with three colors in one orientation, which would mean X would have 3^6 elements, and G be the rotation group of a cube. Therefore, when G acts on X, two elements would be in the same orbit if one is a rotation of the other.

We know that $|G|=24$, so now all we have to count is the number of elements g fixes in X for all $g\in G$.

- One e fixes 3^6 elements
- Six $\frac{\pi}{2}$ face rotations fix 3^3 elements
- Three π face rotations fix 3^4 elements

- Eight $\frac{2\pi}{3}$ vertex rotations fix 3^2 elements
- Six π edge rotations fix 3^3 elements

By using Burnside's lemma, we can find the number of orbits.
$\frac{1}{24}(3^6 + 6 \cdot 3^3 + 3 \cdot 3^4 + 8 \cdot 3^2 + 6 \cdot 3^3) = 57$

2 Pólya Enumeration Theorem

2.1 Simplified Version

Let G be a permutation group of X, and let Y be a finite set of labels. Suppose Z is the set of functions $X \to Y$. Then,

$$|Z/G| = \frac{1}{|G|} \sum_{g \in G} |Y|^{c(g)}$$

where $|Z/G|$ is the number of orbits of Z; $c(g)$ is the number of cycles of g a permutation of X.

This simplified version basically follows from Burnside's Lemma, which states that the number of orbits is equal to the average number of elements fixed by an element of G.

2.2 Cycle Index

Let S be a finite set with m elements, i. e. $|S| = m$. If a permutation ϕ of elements in S splits S into a_1 cycles of length 1, a_2 cycles of length 2, ..., a_m cycles of length m, then the **type** of ϕ is $\{a_1, a_2, \dots, a_m\}$.

Let G be a group of permutations on a finite set S, where $|S| = m$. The polynomial

$$P_G(x_1, x_2, \dots, x_m) = \frac{1}{|G|} \sum_{\phi \in G} x_1^{a_1} x_2^{a_2} \dots x_m^{a_m}$$

in m variables $x_1, x_2, \dots, x_m$, where $\{a_1, a_2, \dots, a_m\}$ is the **cycle index** of ϕ.

2.3 Weights

Let D and R be finite sets, and denote the set of all functions from D to R as R^D. Given a permutation group G of elements of D, two functions f_1, f_2 are equivalent if there exists $g \in G$ such that

$$f_1(g(d)) = f_2(d)$$

It is easy to see that this is an equivalence relation. R^D splits into equivalence class, which are called patterns.

Suppose every element $r \in R$ is assigned a weight. Then the weight $W(f)$ of a function $f \in R^D$ is

$$W(f) = \prod_{d \in D} w[f(d)]$$

2.4 Weighted Version of Pólya's Enumeration Theorem

The number of colorings is given by adding up the coefficient of

$$P_G\{\sum_{r \in R} [w(r)], \sum_{r \in R} [w(r)]^2 \dots\}$$

where P_G is the cycle index. Note that the unweighted version is the case where all of the weights are equal. We will now present a proof by example.

3 Examples

3.1 Chemistry

A major application of this theorem is chemical isomer enumeration. An isomer is a molecule that has the same number of each type of atom but different bonds and, therefore, different structural formulas. We will be looking at benzene's derivatives which are formed by replacing the hydrogen with chlorine. Since the molecule does not change when rotated or flipped, we will the group G is the dihedral group D_6. We get twelve permutations, which we counted in the same way as the first example problem.

We get a function by using cycle index, as shown below:

$$P_{D_6} = \frac{1}{12}(x_1^6 + 2x_3^2 + 4x_2^3 + 3x_1^2x_2^2 + 2x_6)$$

For dichlorobenzene, we will substitute $x_i = (H^i + Cl^i)$ for $i = 1, 2, \ldots, 6$ to get

$$P_{D_6} = \tfrac{1}{12}((H + Cl)^6 + 2(H^3 + Cl^3)^2 + 4(H^2 + Cl^2)^3 + 3(H + Cl)^2(H^2 + Cl^2)^2 + 2(H^6 + Cl^6)) = H^6 + H^5Cl + 3H^4Cl^2 + 3H^3Cl^3 + 3H^2Cl^4 + HCl^5 + Cl^6$$

Thus, there are 12 distinct isomerizations of dichlorobenzene, not counting the original molecule.