The Jacobi ϑ Function

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1 Motivation

Suppose we want to define a nonconstant entire function $f(z)$ on $\mathbb{C}$ which is invariant with respect to a lattice Λ generated by 1 and τ, where $\tau \in \mathbb{C} \setminus \mathbb{R}$. Because of Liouville's Theorem, this is impossible, but we can try to find a nonconstant function that comes as close as possible to being doubly periodic and still entire. With this in mind, we look for quasi-periodic entire functions $f(z)$ satisfying

$$f(z+1) = f(z), \qquad f(z+\tau) = e^{az+b} f(z).$$

Suppose that such a function were to exist. Since f is entire and periodic, we can write f in terms of a Fourier series:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z},$$

where the a_n's are the Fourier coefficients of f. Let us evaluate $f(z+\tau+1)$ in two ways. Let us first absorb the 1 and then the τ to get

$$f(z+\tau+1) = f(z+\tau) = e^{az+b} f(z). \tag{1}$$

If we absorb the τ before the 1, we get

$$f(z+\tau+1) = e^{a(z+1)+b} f(z+1) = e^a e^{az+b} f(z). \tag{2}$$

Therefore $a = 2\pi ik$ for some $k \in \mathbb{Z}$. Now let us substitute the Fourier series for $f(z+\tau)$ into (1) to get

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} a_n e^{2\pi in\tau} e^{2\pi inz} &= f(z+\tau) \\
&= e^{2\pi ikz+b} f(z) \\
&= \sum_{n=-\infty}^{\infty} a_n e^{2\pi i(n+k)z+b} \\
&= \sum_{n=-\infty}^{\infty} a_{n-k} e^{2\pi inz+b}.
\end{aligned}$$

Therefore

$$a_n = a_{n-k} e^{b-2\pi in\tau}$$

for all $n \in \mathbb{Z}$. If $k = 0$, we have the rather dull solution of $f(z) = e^{2\pi iz}$. If, however, we choose $k = -1$, we have, for all $n \in \mathbb{Z}$,

$$a_n = a_0 e^{\pi in(n-1)\tau - nb}.$$

Therefore

$$f(z) = a_0 \sum_{n=-\infty}^{\infty} e^{\pi in^2\tau + 2\pi inz - \pi in\tau - nb}.$$

With this in mind, we define a function ϑ of two variables:

$$\vartheta(z,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi in^2\tau + 2\pi inz},$$

where $z \in \mathbb{C}$ and $\tau \in \mathbb{H} = \{w \in \mathbb{C} \mid \Im w > 0\}$.

One motivation for working with the ϑ function is that it provides us with very elegant solutions to several problems in number theory, as we shall soon see.

2 Elliptic Properties of the ϑ Function

As mentioned in the previous section, we define $\vartheta : \mathbb{C} \times \mathbb{H} \to \mathbb{C}$ by

$$\vartheta(z,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi in^2\tau + 2\pi inz}.$$

Note that the series converges absolutely and uniformly on all compact sets. If we hold τ constant, then the series looks like a Fourier series, so we can write

$$\vartheta(z,\tau) = \sum_{n=-\infty}^{\infty} a_n(\tau) e^{2\pi i n z},$$

where $a_n(\tau) = e^{\pi i n^2 \tau}$. It is clear that ϑ has period 1 in the first variable, but the Fourier coefficients suggest some other quasi-periodic behavior with a quasi-period of τ. This is in fact the case:

$$\begin{aligned}
\vartheta(z+\tau,\tau) &= \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau + 2\pi i n(z+\tau)} \\
&= \sum_{n=-\infty}^{\infty} e^{\pi i (n+1)^2 \tau - \pi i \tau + 2\pi i n z} \\
&= \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau - \pi i \tau + 2\pi i n z - 2\pi i z} \\
&= e^{-\pi i \tau - 2\pi i z} \vartheta(z,\tau).
\end{aligned}$$

This indeed corresponds to the $k = -1$ case mentioned in the previous section when we restrict our function to the first variable. Therefore the ϑ function exhibits properties similar to those of elliptic functions. In fact, after finding a product formula for $\vartheta(z,\tau)$, we will find an elliptic function related to both the ϑ function and the Weierstraß $\wp$ function.

Before we work out the product formula for the ϑ function, let us note that $e^{\pi i \tau}$ is frequently denoted by q, with the understanding that the normally multivalued q^w will denote $e^{\pi i w \tau}$. With this in mind, our ϑ function becomes

$$\vartheta(z,q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2\pi i n z},$$

defined on $\mathbb{C} \times \mathbb{D}^*$, where $\mathbb{D}^* = \{w \in \mathbb{C} : 0 < |w| < 1\}$. We shall adopt this convention when convenient from now on.

As is often the case in mathematics and especially in complex analysis, we would like to find a product representation for the ϑ function. Therefore it is useful to look for the zeros of $\vartheta(z,\tau)$. We shall therefore show that if $z = \left(m+\frac{1}{2}\right) + \left(n+\frac{1}{2}\right)\tau$ for some $m,n \in \mathbb{Z}$, then $\vartheta(z,\tau) = 0$. Because of the quasi-periodic properties, it suffices to show that $\vartheta\left(\frac{1+\tau}{2},\tau\right) = 0$. Therefore we write

$$\vartheta\left(\frac{1+\tau}{2},\tau\right) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i (n^2+n)\tau}.$$

Since the sum is absolutely convergent, we can change the order of the terms. Therefore we sum n and $-n-1$ together and notice that the two of these sum to 0. Therefore the entire sum is 0, as desired.

Let us define a product that has simple zeros at the same places as the zeros of ϑ that we already know about. An appropriate definition would therefore be

$$\Pi(z,\tau) = \prod_{n=1}^{\infty} (1-q^{2n})(1+q^{2n-1}e^{2\pi i z})(1+q^{2n-1}e^{-2\pi i z}),$$

which is holomorphic for z and τ in $\mathbb{C}\times\mathbb{H}$. Let us see if the Π function satisfies similar properties to those of the ϑ function. First, it is clear that $\Pi(z+1,\tau) = \Pi(z,\tau)$ because of the $2\pi i$-periodicity of the exponential function. Now we work out $\Pi(z+\tau,\tau)$. We have

$$\begin{aligned}
\Pi(z+\tau,\tau) &= \prod_{n=1}^{\infty} (1-q^{2n})(1+q^{2n+1}e^{2\pi i z})(1+q^{2n-3}e^{-2\pi i z}) \\
&= \left(\frac{1+q^{-1}e^{-2\pi i z}}{1+qe^{2\pi i z}}\right)\Pi(z,\tau) \\
&= \Pi(z,\tau) q e^{-2\pi i z} \\
&= \Pi(z,\tau) e^{-\pi i \tau - 2\pi i z},
\end{aligned}$$

also just like ϑ.

Since $\vartheta(z,\tau)$ has zeros at the same places as $\Pi(z,\tau)$ (and perhaps some others), the function $\vartheta(z,\tau)/\Pi(z,\tau)$ is an entire and bounded (because it is elliptic) function in z. Therefore it must be independent of z. Therefore we can write

$$\vartheta(z,\tau) = c(\tau)\Pi(z,\tau). \tag{3}$$

We now show that $c(\tau) = c(4\tau)$ for all τ. If we let $z = \frac{1}{2}$ in (3), we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} &= c(\tau) \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{2n-1})(1 - q^{2n-1}) \\ &= c(\tau) \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{2n-1}).\end{aligned}$$

Therefore we have

$$c(\tau) = \frac{\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}}{\prod_{n=1}^{\infty} (1 - q^n)(1 - q^{2n-1})}.$$

If we now let $z = \frac{1}{4}$ in (3), then we have

$$\begin{aligned}\vartheta\left(\frac{1}{4}, \tau\right) &= \sum_{n=-\infty}^{\infty} q^{n^2} i^n \\ &= \sum_{n=-\infty}^{\infty} (-1)^n q^{4n^2}.\end{aligned}$$

We also have

$$\begin{aligned}\Pi\left(\frac{1}{4}, \tau\right) &= \prod_{n=1}^{\infty} (1 - q^{2n})(1 + iq^{2n-1})(1 - iq^{2n-1}) \\ &= \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{4n-2}) \\ &= \prod_{n=1}^{\infty} (1 - q^{4n})(1 - q^{8n-4}).\end{aligned}$$

Thus we have

$$c(\tau) = \frac{\sum_{n=-\infty}^{\infty} (-1)^n q^{4n^2}}{\prod_{n=1}^{\infty} (1 - q^{4n})(1 - q^{8n-4})}.$$

When we equate our two expressions for $c(\tau)$, we see that $c(\tau) = c(4\tau)$. Thus for any $k \in \mathbb{Z}$, we have $c(\tau) = c(4^k\tau)$. Since $e^{4^k \pi i \tau}$ goes to 0 as k goes to ∞, we can put this back into either expression for $c(\tau)$ and notice that in the limit we are left with the term corresponding to $n = 0$ in the sum, and the denominator goes to 1. Therefore $c(\tau)$ is identically 1. Thus we have proven the following theorem:

Theorem. *For all $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$, we have*

$$\vartheta(z,\tau) = \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}e^{2\pi i z})(1+q^{2n-1}e^{-2\pi i z}).$$

The product representation is known as the Jacobi Triple Product. As we well see later on, it leads quickly to important applications in the theory of integer partitions. Also, it tells us that the zeros of $\vartheta(z,\tau)$ are *exactly* $\left\{\left(\frac{1}{2}+m\right)+\left(\frac{1}{2}+n\right) \mid m,n \in \mathbb{Z}\right\}$.

Now that we know the zeros of $\vartheta(z,\tau)$, we note that

$$\varsigma(z,\tau) = \frac{\partial^2}{\partial z^2}(\log \vartheta(z,t))$$

is an elliptic function in z with lattice $\Lambda = \{m+n\tau \mid m,n \in \mathbb{Z}\}$. It is easy to see that ς has one double pole in $\mathbb{C}/\Lambda$ at $z = \frac{1}{2}+\frac{\tau}{2}$. Therefore we know that $\varsigma(z,\tau)$ can be written in terms of the Weierstraß $\wp$ function. As it turns out, we have the identity

$$\varsigma(z,\tau) = \wp\left(z - \frac{1}{2} - \frac{\tau}{2}\right) + C,$$

where C depends on τ but not on z.

3 Modular Properties of the ϑ Function

As we saw in the previous section, the first variable of the ϑ function makes it act like an elliptic function. The second variable makes it act more like a modular function, as we shall see in this section. To this end, we shall fix $z=0$ and treat the ϑ function as a function in just one variable, namely τ. We shall prove the following result:

Theorem. *For any $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, with ab and cd even, we have*

$$\vartheta(0,\gamma(\tau)) = \zeta\sqrt{c\tau+d}\,\vartheta(0,\tau), \tag{4}$$

where as usual $\gamma(\tau) = \frac{a\tau+b}{c\tau+d}$ and ζ is a particular 8^{th} root of unity.

Proof. First let us note that the center of $\mathrm{SL}_2(\mathbb{Z})$, namely $\pm I$, acts trivially on the ϑ function, so we may assume without loss of generality that $c \geq 0$. Therefore $\Im(c\tau + d) \geq 0$, and we select $\sqrt{c\tau + d}$ so that it has nonnegative real and imaginary parts. If we wish to choose the other square root, then we can do so, as it will only change ζ by a factor of -1, so it will still be an 8^{th} root of unity. We now proceed by induction on $|c| + |d|$, so we first check the base cases $\gamma = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$, where b is even, and $\gamma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. In the first case, we have

$$\vartheta(0, \tau + b) = \vartheta(0, \tau) \tag{5}$$

just by looking at the definition of ϑ. For the second case, we consider the function $f(x) = e^{-\pi x^2 t}$ and notice that its Fourier transform is

$$\widehat{f}(x) = \int_{-\infty}^{\infty} e^{-\pi \xi^2 t} e^{-2\pi i \xi x} \, d\xi = \frac{1}{\sqrt{t}} e^{-\pi x^2 / t}.$$

Then by the Poisson Summation Formula we have

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n),$$

or

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2 / t}.$$

Now we notice that this result simply states that

$$\vartheta(0, i\tau) = \frac{1}{\sqrt{\tau}} \vartheta\left(0, \frac{i}{\tau}\right),$$

or, after a change of variables,

$$\vartheta\left(0, -\frac{1}{\tau}\right) = e^{-i\pi/4} \sqrt{\tau} \vartheta(0, \tau). \tag{6}$$

Therefore the base cases hold. For the inductive step, first suppose that $|d| > |c|$. By (5), we can replace τ in (4) with $\tau \pm 2$ so that (4) for a, b, c, d follows from that of $a, b \pm 2a, c, d \pm 2c$. Then one of $|d \pm 2c| < |d|$, so in this case our inductive step holds. In the other case, namely $|d| < |c|$, we replace τ by $-\frac{1}{\tau}$ and use (6) so that (4) for a, b, c, d follows from that of $b, -a, d, -c$ after some computations, which reduces us to the $|d| > |c|$ case. In either case, we have reduced $|c| + |d|$, so the inductive step holds.

In fact, if we are a bit careful about keeping track of the values we are working with, we can compute ζ exactly. If c is even and d odd, then we have

$$\zeta = i^{(d-1)/2}\left(\frac{c}{|d|}\right),$$

where $\left(\frac{m}{n}\right)$ is the Jacobi symbol for quadratic residues, and if c is odd and d even, we have

$$\zeta = e^{-\pi i c/4}\left(\frac{d}{c}\right).$$

In fact, our theorem generalizes to the case in which the first variable is nonzero. The more general theorem is

$$\vartheta\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = \zeta\sqrt{c\tau+d}\,e^{\pi i c z^2/(c\tau+d)}\vartheta(z,\tau),$$

where ζ is the same as before. ■

To study modular properties of the ϑ function further, it is useful to define four auxiliary functions:

$$\begin{array}{rclcl}
\vartheta_{00}(z,\tau) & = & \sum_{n=-\infty}^{\infty} e^{\pi i n^2\tau+2\pi i n z} & = & \vartheta(z,\tau) \\
\vartheta_{01}(z,\tau) & = & \sum_{n=-\infty}^{\infty} e^{\pi i n^2\tau+2\pi i n(z+1/2)} & = & \vartheta\left(z+\frac{1}{2},\tau\right) \\
\vartheta_{10}(z,\tau) & = & \sum_{n=-\infty}^{\infty} e^{\pi i(n+1/2)^2\tau+2\pi i(n+1/2)z} & = & e^{\pi i\tau/4+\pi i(z+1/2)}\vartheta\left(z+\frac{\tau}{2},\tau\right) \\
\vartheta_{11}(z,\tau) & = & \sum_{n=-\infty}^{\infty} e^{\pi i(n+1/2)^2\tau+2\pi i(n+1/2)(z+1/2)} & = & e^{\pi i\tau/4+\pi i(z+1/2)}\vartheta\left(z+\frac{1+\tau}{2},\tau\right).
\end{array}$$

We notice the following modular properties of the auxiliary ϑ functions:

$$\begin{array}{rclrcl}
\vartheta_{00}(z,\tau+1) & = & \vartheta_{01}(z,\tau) & \vartheta_{00}\left(\frac{z}{\tau},-\frac{1}{\tau}\right) & = & \sqrt{-i\tau}\,e^{\pi i z^2/\tau}\vartheta_{00}(z,\tau) \\
\vartheta_{01}(z,\tau+1) & = & \vartheta_{00}(z,\tau) & \vartheta_{01}\left(\frac{z}{\tau},-\frac{1}{\tau}\right) & = & \sqrt{-i\tau}\,e^{\pi i z^2/\tau}\vartheta_{10}(z,\tau) \\
\vartheta_{10}(z,\tau+1) & = & e^{\pi i/4}\vartheta_{10}(z,\tau) & \vartheta_{10}\left(\frac{z}{\tau},-\frac{1}{\tau}\right) & = & \sqrt{-i\tau}\,e^{\pi i z^2/\tau}\vartheta_{01}(z,\tau) \\
\vartheta_{11}(z,\tau+1) & = & e^{\pi i/4}\vartheta_{11}(z,\tau) & \vartheta_{11}\left(\frac{z}{\tau},-\frac{1}{\tau}\right) & = & -\sqrt{-i\tau}\,e^{\pi i z^2/\tau}\vartheta_{11}(z,\tau).
\end{array}$$

We now recall the following definitions:

Definition. *The level N principal congruence subgroup, Γ_N, of $\mathrm{SL}_2(\mathbb{Z})$ is*

$$\Gamma_N = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z} \mid b, c \equiv 0 \pmod{N},\ a, d \equiv 1 \pmod{N} \right\}.$$

Definition. *If $k \in \mathbb{Z}_{\geq 0}$ and $N \in \mathbb{N}$, we call a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ a modular form of weight k and level N if*

(a) for all $\tau \in \mathbb{H}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_N$,

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau),$$

(b) the Fourier expansion of f has the form

$$f(\tau) = \sum_{n=0}^{\infty} \alpha_n e^{2\pi i n \tau}$$

(i.e. all the α_{-n} are 0).

We can now give a useful result on modular forms, but we shall not prove it here.

Proposition. *The functions $\vartheta_{00}^2(0, \tau)$, $\vartheta_{01}^2(0, \tau)$, and $\vartheta_{10}^2(0, \tau)$ are modular forms of weight 1 and level 4.*

This fact allows us to find a holomorphic map from $\mathbb{H}/\Gamma_4$ into complex projective 2-space, which we shall denote by $\mathbb{P}^2$. We shall call this map $\Psi_2 : \mathbb{H}/\Gamma_4 \to \mathbb{P}^2$, and it is given by

$$\Psi_2(\tau) = (\vartheta_{00}^2(0, \tau), \vartheta_{01}^2(0, \tau), \vartheta_{10}^2(0, \tau))$$

since none of the $\vartheta_{ij}(0, \tau)$ are ever zero. As usual, we want to see what happens when we let $\tau \mapsto \tau + 1$ and $\tau \mapsto -\frac{1}{\tau}$. Let $\Psi_2(\tau) = (\alpha, \beta, \gamma)$. Then we have

$$\Psi_2(\tau + 1) = (\beta, \alpha, i\gamma) \qquad \Psi_2\left(-\frac{1}{\tau}\right) = (\alpha, \gamma, \beta).$$

Therefore the image of Ψ_2 is the complex projective cone

$$\alpha^2 = \beta^2 + \gamma^2$$

(which is essentially just a nondegenerate conic section in $\mathbb{C}^3$) without the points $(1, 0 \pm 1), (1, \pm 1, 0), (0, 1, \pm i)$. We would like to be able to extend the image of Ψ_2 onto the cone. To do this, we note that as $\Im\tau \to \infty$,

$$\vartheta_{00}(0,\tau) \to 1, \qquad \vartheta_{01}(0,\tau) \to 1, \qquad \vartheta_{10}(0,\tau) \to 0.$$

Thus $\Psi_2(\tau) \to (1,1,0)$. The action of $\mathrm{SL}_2(\mathbb{Z})$ on $(1,1,0)$ will give us the rest of the missing points. Therefore we consider the compactification of $\mathbb{H}/\Gamma_4$, which we shall call Ξ. This extends Ψ_2 to a holomorphic map for all of Ξ.

Notice that in the mapping properties of the ϑ_{ij} functions above, we do not use ϑ_{11} because $\vartheta_{11}(0,\tau) \equiv 0$, and so this function is uninteresting. However, the derivative of the ϑ_{11} function with respect to z at $z = 0$ is much more interesting. Let us abbreviate

$$\frac{\partial}{\partial z}\vartheta_{11}(z,\tau)\bigg|_{z=0} = \vartheta_{11}'(0,\tau).$$

We shall not prove it here, but we have the following result, known as the Jacobi Derivative Formula:

Proposition. *The auxiliary ϑ functions satisfy*

$$\vartheta_{11}'(0,\tau) = -\pi\vartheta_{00}(0,\tau)\vartheta_{01}(0,\tau)\vartheta_{10}(0,\tau)$$

for all $\tau \in \mathbb{H}$.

4 A Result in Combinatorics

By a *strong partition* of a nonnegative integer n, we mean a k-tuple $a = (a_1, \ldots, a_k) \in \mathbb{N}^k$ for which $a_1 > \cdots > a_k > 0$ and $a_1 + \cdots + a_k = n$. If k is even, we shall call the strong partition a an even strong partition, and if k is odd, we shall call the strong partition an odd strong partition. Let $\alpha(n)$ denote the number of even strong partitions of n and $\beta(n)$ the number of odd strong partitions. We aim to find a relationship between $\alpha(n)$ and $\beta(n)$. To this end, we first notice that the generating function for $\alpha(n) - \beta(n)$ is

$$\sum_{n=0}^{\infty} (\alpha(n) - \beta(n))\, x^n = \prod_{n=1}^{\infty} (1 - x^n)$$

for $x \in \mathbb{D} = \{w \in \mathbb{C} : |w| < 1\}$. Now select $x = e^{2\pi i u}$, where $u \in \mathbb{H}$, so that we have

$$\prod_{n=1}^{\infty}(1 - x^n) = \prod_{n=1}^{\infty}(1 - e^{2\pi i n u}).$$

This is a special case of the Jacobi Triple Product by writing $q = e^{3\pi i u}$ and $z = \frac{1}{2} + \frac{1}{2}u$. This gives us

$$\prod_{n=1}^{\infty}(1 - q^{2n})(1 + q^{2n-1}e^{2\pi i z})(1 + q^{2n-1}e^{-2\pi i z}).$$

By the Jacobi Triple Product, we therefore have

$$\begin{aligned}
\prod_{n=1}^{\infty}(1 - x^n) &= \prod_{n=1}^{\infty}(1 - e^{2\pi i n u}) \\
&= \sum_{n=-\infty}^{\infty}(-1)^n e^{3\pi i n^2 u + \pi i n u} \\
&= \sum_{n=-\infty}^{\infty}(-1)^n e^{(3n^2+n)\pi i u} \\
&= \sum_{n=-\infty}^{\infty}(-1)^n x^{(3n^2+n)/2}.
\end{aligned}$$

Therefore we have derived the remarkable identity, first proven by Euler, that

$$\alpha(n) - \beta(n) = \begin{cases} (-1)^k & n = \frac{3k^2+k}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

5 Results in Number Theory

By virtue of the n^2 terms in the exponents in the definition of $\vartheta(z, \tau)$, we find the Jacobi ϑ function very useful for solving problems dealing with sums of squares. To this effect, define a representation function $r_h : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ by

$$r_h(n) = \#\{(x_1, \ldots, x_h) \in \mathbb{Z}^h \mid x_1^2 + \cdots + x_h^2 = n\}.$$

Using the ϑ function, we will find explicit formulae for $r_2(n)$ and $r_4(n)$.

5.1 The Two Squares Theorem

Let us denote by $d_1(n)$ the number of divisors d of n with $d \equiv 1 \pmod 4$ and $d_3(n)$ the number of divisors d of n with $d \equiv 3 \pmod 4$. Notice that $d_1(n) \geq d_3(n)$. We shall prove the following result:

Theorem. *If $n \in \mathbb{N}$, then $r_2(n) = 4(d_1(n) - d_3(n))$.*

Proof. The first thing that we need to notice is that $\vartheta(0,\tau)^2$ is the generating function for $r_2(n)$. To see, this, we write

$$\begin{aligned}
\vartheta(0,\tau)^2 &= \left(\sum_{m=-\infty}^{\infty} q^{m^2}\right)\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right) \\
&= \sum_{m,n\in\mathbb{Z}} q^{m^2+n^2} \\
&= \sum_{n=0}^{\infty} r_2(n) q^n.
\end{aligned}$$

Now we reduce the problem to showing a different identity, namely

$$\vartheta(0,\tau)^2 = 2\sum_{n=-\infty}^{\infty} \frac{1}{q^n + q^{-n}}. \tag{7}$$

To see this, note that the right side of (7) is equal to $1 + 4\sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}}$ and that

$$\frac{1}{1+q^{2n}} = \frac{1-q^{2n}}{1-q^{4n}},$$

so that the right side of (7) becomes

$$1 + 4\sum_{n=1}^{\infty}\left(\frac{q^n}{1-q^{4n}} - \frac{q^{3n}}{1-q^{4n}}\right).$$

But this is clearly the generating function for $4(d_1(n) - d_3(n))$ (except for the $n = 0$ case, which we can ignore anyway since the theorem assumes $n \geq 1$). Let us now write

$$C(\tau) = 2\sum_{n=1}^{\infty} \frac{1}{q^n + q^{-n}} = \sum_{n=-\infty}^{\infty} \frac{1}{\cos(\pi n \tau)}.$$

Therefore we wish to show that $\vartheta(0,\tau)^2 = C(\tau)$.

We now show that $C(\tau)$ and $\vartheta(0,\tau)^2$ satisfy the same sorts of properties under transformations of the modular group $\mathrm{SL}_2(\mathbb{Z})$. By (6), we have

$$\vartheta(0,\tau)^2 = \frac{i}{\tau}\vartheta\left(-\frac{1}{\tau}\right)^2,$$

so now we shall show that the same law property holds for $C(\tau)$. To show this, we use the Poisson Summation Formula on

$$f(x) = \frac{e^{-2\pi iax}}{\cosh(\pi x/\tau)}$$

and set $a = 0$ to get

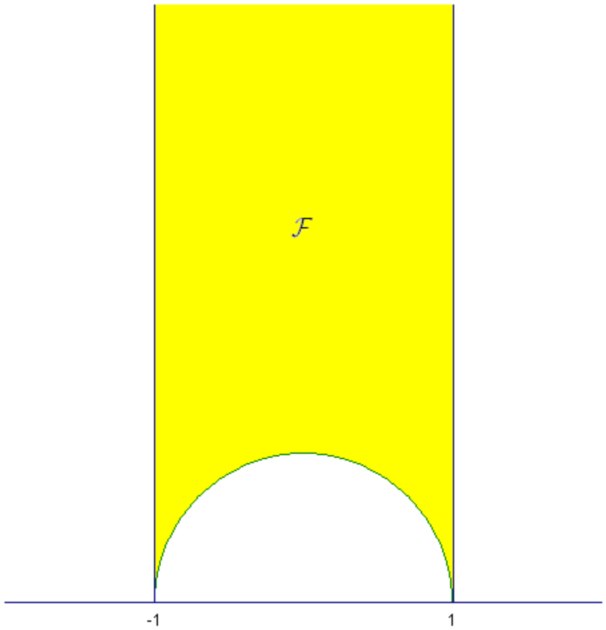
$$\sum_{n=-\infty}^{\infty} \frac{1}{\cosh(\pi n/\tau)} = \sum_{n=-\infty}^{\infty} \frac{\tau}{\cosh(\pi n\tau)},$$

or, equivalently,

$$C(\tau) = \frac{i}{\tau}C\left(-\frac{1}{\tau}\right),$$

as desired. Also, we clearly have $\vartheta(0,\tau+2)^2 = \vartheta(0,\tau)^2$ and $C(\tau+2) = C(\tau)$. We can also easily show that $\vartheta(0,\tau)^2 \to 1$ and $C(\tau) \to 1$ as $\Im\tau \to \infty$ and that $\vartheta(0,1-1/\tau)^2 \sim 4\frac{\tau}{i}e^{\pi i\tau/2}$ and $C(1-1/\tau) \sim 4\frac{\tau}{i}e^{\pi i\tau/2}$ as $\Im\tau \to \infty$ with just a few computations.

In other words, we are looking at these two functions on the fundamental domain $\mathcal{F} = \{\tau \in \mathbb{H} \cup \mathbb{R} : |\Re\tau| \leq 1, |\tau| \geq 1\}$.



We now have a function $f(\tau) = C(\tau)/\vartheta(0,\tau)^2$, defined for $\tau \in \mathbb{H}$, satisfying $f(\tau+2) = f(\tau)$, $f(-1/\tau) = f(\tau)$, and $f(\tau)$ is bounded. We will show that any such function is constant. Note that the points at ± 1 are in $\mathcal{F}$ but not in the domain of f. We call these points cusps. A key step in the proof of the Two Squares Theorem will be determining the behavior of f near the cusps.

Notice that every point in $\mathbb{H}$ can be mapped onto $\mathcal{F}$ by some finite sequence of transformations $T : \tau \to \tau + 2$ and $S : \tau \to -\frac{1}{\tau}$ or their inverses, we can understand f completely by understanding it on $\mathcal{F}$. More precisely, let $G = \langle T, S \rangle$, and let G act on $\mathbb{H}$ by writing an element of G as $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so that $g(\tau) = \frac{a\tau+b}{c\tau+d}$. Notice that G is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Also, we have $\Im g(\tau) = \frac{\Im \tau}{|c\tau+d|^2}$.

Let us now turn again to proving $f \equiv C(\tau)/\vartheta(0,\tau)^2$ constant. Suppose, on the contrary, that f were nonconstant, and write $f(\tau) = g(z)$, where $z = e^{\pi i \tau}$ so that g is defined on the punctured disc. Since f is bounded, 0 must be a removable singularity, and so we set $g(0) = \lim_{\Im\tau\to\infty} f(\tau)$. Then by the Maximum Modulus Principle,

$$\lim_{\Im\tau\to\infty} |f(\tau)| < \sup_{\tau\in\mathcal{F}} |f(\tau)|.$$

Now we investigate the behavior of f near the cusps. We need only consider the cusp at 1 since the two are equivalent. Let us define

$$U_n = \begin{bmatrix} 1-n & n \\ -n & 1+n \end{bmatrix},$$

$$T_n(\tau) = \tau + n,$$

and $\mu(\tau) = \frac{1}{1-\tau}$. Then we have

$$U_n = \mu^{-1} T_n \mu,$$

so

$$U_n U_m = U_{n+m}$$

and $U_{-1} = TS$. Thus $U_n \in G$ for each n. Therefore we have $f(\mu^{-1} T_n \mu(\tau)) = f(\tau)$, so $F(\tau) = f(\mu^{-1}(\tau))$ has period 1, i.e.

$$F(T_n \tau) = F(\tau)$$

for all $n \in \mathbb{Z}$. If we now write $h(z) = F(\tau)$, where $z = e^{2\pi i \tau}$, then h has a removable singularity at 0, so we have

$$\lim_{\Im\tau \to \infty} \left| f\left(1 - \frac{1}{\tau}\right) \right| < \sup_{\tau \in \mathcal{F}} |f(\tau)|.$$

Thus f attains a maximum in $\mathbb{H}$, contradicting the Maximum Modulus Principle. Thus f must be constant. Since $f \to 1$ at the cusps, f must be identically 1, and the Two Squares Theorem is proven. ■

Corollary. *A positive integer can be represented as the sum of two squares if and only if every prime factor congruent to* 3 (mod 4) *is raised to an even power.*

5.2 The Four Squares Theorem

As in the two squares theorem, we have a simple way to express the number of representations of n as the sum of four squares. However, the proof for this theorem is somewhat more complicated than the two squares theorem and relies on Eisenstein series. We state it without proof here:

Theorem. *If* $n \in \mathbb{N}$, *then*

$$r_4(n) = 8 \sum_{\substack{d|n \\ 4\nmid d}} d.$$

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