DIFFERENTIAL POSETS

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Abstract. In this paper, we give a sampling of the theory of differential posets, including various topics that excited me. Most of the material is taken from Richard Stanley's survey paper [3], and some is from the section on differential posets in the second edition of his book [2]. Not too much is original: I reworked a few of the proofs, so they might be a bit different from Stanley's version, and some of the material of random walks is my own contribution.

1. Introduction and Examples

We begin, appropriately enough, by defining differential posets.

Definition 1. Let r be a positive integer. We say that a poset P is r-differential if

(1) P is locally finite, graded, and has a $\widehat{0}$ element.
(2) If x and y are two distinct elements of P, and there are k elements covered by both x and y, then there are also k elements covering both x and y.
(3) If $x \in P$ and x covers exactly k elements of P, then x is covered by exactly $k+r$ elements of P.

If P is r-differential for some r, we say that P is a differential poset.

In fact, part (2) of the definition can only happen for $k=0$ or 1, for if x and y are of minimal rank with the property that they are covered by $k \geq 2$ elements, they also cover k elements, so in particular they cover two elements z_1 and z_2 that are covered by x and y, and hence $k \geq 2$ for z_1 and z_2, which contradicts minimality of rank.

It is not completely trivial to find examples of differential posets. Probably the easiest example is the Young poset Y, defined as follows: As a set, Y consists of all Young tableaux for all partitions of all nonnegative integers. We say that $\lambda \leq \mu$ if λ is contained in μ, i.e., if $\lambda = \lambda_1 + \lambda_2 + \cdots$ and $\mu = \mu_1 + \mu_2 + \cdots$, where the λ_i's and μ_i's are nonincreasing, then $\lambda_i \geq \mu_i$ for all i.

Proposition 2. *Y is 1-differential.*

Proof. First, Y is locally finite because between any two Young tableaux λ and μ, with $\lambda \leq \mu$, there are only finitely many other Young tableaux lying between them. Y is graded as follows: the rank of λ is n if λ is a partition of n. Y has a minimal element which is the (unique) partition of 0. Now, let λ and μ be two different partitions. If

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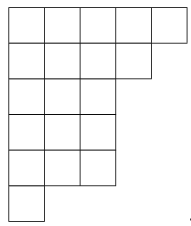
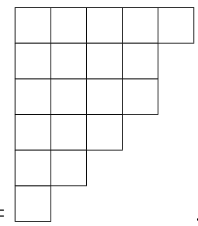
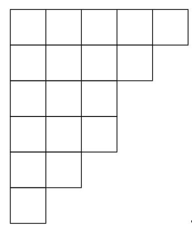
there are any elements covered by both λ and μ, then λ and μ are both partitions of the same integer n, and there's exactly one element covering both of them, namely the union of their diagrams. In this case, they both cover the intersection of the diagrams. Finally, suppose λ covers k elements. Then k is the number of distinct parts of λ, since we can remove one from one of these distinct parts to make a partition λ' covered by λ. The number of partitions covering λ is then $k+1$, for we can increase any part by 1, or we can add a new part of size 1. ■

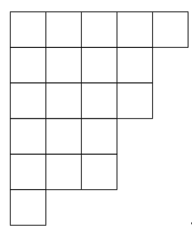
Let's look at an example of the second condition for being a differential poset. Let

$\lambda =$  , $\mu =$  .

They both cover their intersection

 ,

and both are covered by their union

 .

Once we have one differential poset P, we can generate others.

Proposition 3. *If P is an r-differential poset, then the poset P^k is a kr-differential poset.*

Proof. The minimal element is $(\widehat{0}, \ldots, \widehat{0})$. It is graded by the sum of the gradings of the components, and it is locally finite, since an interval is the product of the component intervals. If x and y cover z, then x and y differ in one or two components only. Suppose they differ in exactly one component. Then we can write $x = (a_1, \ldots, a_i, \ldots, a_k)$ and $y = (a_1, \ldots, a_i', \ldots, a_k)$. Thus a_i and a_i' cover a unique element, so they are covered by a unique element a_i''. Thus x and y are both covered by $(a_1, \ldots, a_i'', \ldots, a_k)$. Now, suppose they differ in exactly two places, so that

$x = (a_1, \ldots, a_i, \ldots, a_j, \ldots, a_k)$ and $y = (a_1, \ldots, a_i', \ldots, a_j', \ldots, a_k)$. Thus either a_i covers a_i' and a_j is covered by a_j', or the reverse is true. Without loss of generality, suppose a_i covers a_i' and a_j is covered by a_j'. Then x and y cover $(a_1, \ldots, a_i', \ldots, a_j, \ldots, a_k)$ and are covered by $(a_1, \ldots, a_i, \ldots, a_j', \ldots, a_k)$. Finally, suppose x covers ℓ elements, and that the i^{th} component of x covers ℓ_i elements, so that $\sum \ell_i = \ell$. Then the i^{th} component of x is covered by $\ell_i + r$ elements, so x is covered by $\sum(\ell_i + r) = \ell + kr$ elements, as desired. ■

Since Y is a 1-differential poset, then, Y^r is an r-differential poset. Hence, r-differential posets exist for all r.

There is another class of important differential posets, the so-called Fibonacci posets.

Definition 4. Let $A_r = \{1_1, 1_2, \ldots, 1_r, 2\}$, and let A_r^* be the set of all finite-length words in the elements of A_r. We define a poset $Z(r)$ to be the poset whose underlying set is A_r^*, and so that if $w_1, w_2 \in Z(r)$, then w_1 covers w_2 if either

(1) we obtain w_2 from w_1 by changing the last 2 in the initial string of 2's in w_1 to some 1_i, or
(2) we obtain w_2 from w_1 by deleting the first 1_i from w_1.

We call $Z(r)$ the r-Fibonacci poset.

Theorem 5. *$Z(r)$ is an r-differential poset.*

Proof. The minimal element $\widehat{0}$ is the empty word, and $Z(r)$ is graded by the sum of the weights of the letters in its string, where the weight of 1_i is 1 and the weight of 2 is 2. Since there are only finitely many possible strings of each grading and the grading is discrete, $Z(r)$ is locally finite.

Let's now check condition (2). Suppose x and y are two elements of $Z(r)$ so that x and y cover a unique element z. Let $z = 2^k 1_i s$ for some string s. (It's possible there is no 1_i; that will change very little.) We have two cases now:

(1) $x = 2^{k+1}s$ and $y = 2^\ell 1_j 2^{k-\ell} 1_i s$,
(2) $x = 2^m 1_p 2^{k-m} 1_i s$ and $y = 2^\ell 1_j 2^{k-\ell} 1_i s$.

In either case, we take $w = 2^{k+1} 1_i s$. Then w is the the unique element covering both x and y. Similarly, if there's an element covering x and y, we can construct an element covered by both x and y.

Finally, we check condition (3). If the initial string of 2's of x has length k, then we could have obtained this initial string by replacing any 1_i in any of these positions with a 2, in kr ways; if the initial string is followed by some 1_i, then there's one more way we could have obtained x, namely from inserting this 1_i. Hence there are either kr or $kr + 1$ elements covered by x, depending on whether the string has any 1's or not. To obtain strings covering x, we can place a 1_i between any two 2's or at the beginning or end of the string of 2's, in $(k+1)r$ ways, and if x has any 1's, we can also replace the initial 1 with a 2. Hence there are either $(k+1)r$ or $(k+1)r+1$ words covering x. Thus $Z(r)$ is r-differential. ∎

In fact, $Z(r)$ is irreducible, in the sense that if $Z(r) = P \times Q$, then either P or Q is the trivial poset.

If P is a graded poset and $x \in P$, write $\rho(x)$ for the rank of x, and write p_n for the number of elements of P of rank n. An interesting open problem on differential posets is the following:

Conjecture 6. *Let P be an r-differential poset. Then for each $i \geq 0$, $p_i(Y^r) \leq p_i(P) \leq p_i(Z(r))$.*

The r-Fibonacci poset is so-named because $p_i(Z(1)) = F_{i+1}$, the $(i+1)^{\text{st}}$ Fibonacci number. More generally, $p_i(Z(r))$ satisfies the recursion

$$p_{i+1}(Z(r)) = rp_i(Z(r)) + p_{i-1}(Z(r)), \qquad p_0(Z(r)) = 1, \qquad p_1(Z(r)) = r.$$

A problem that Richard Stanley posed in his survey paper on differential posets [3] is to determine the automorphism groups of the $Z(r)$'s. He suggested that this would probably not be too difficult, and this turned out to be true: the problem was solved by Jonathan Farley and Sungsoon Kim in 2004 [1]. Here we'll determine the automorphism group only for $Z(1)$, although the general case is similar in flavor, albeit a bit more tedious.

Theorem 7. *The automorphism group of $Z(1)$ is $\mathbb{Z}/2\mathbb{Z}$, with the nontrivial automorphism being defined by*

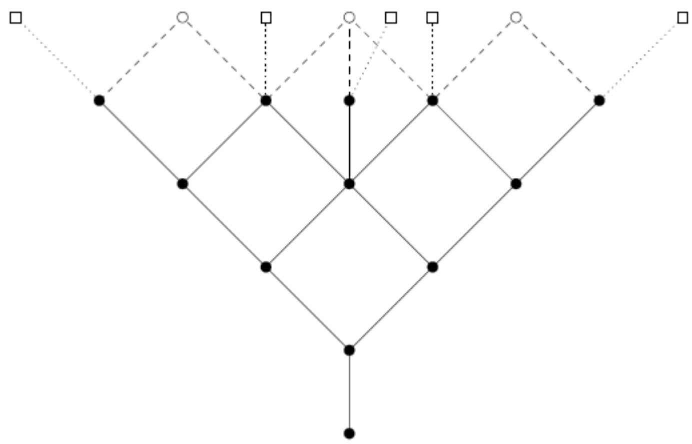
$$\sigma(w11) = w2, \qquad \sigma(w2) = w11, \qquad \sigma(x) = x \text{ for } x \text{ not ending in 11 or 2.}$$

Proof. It suffices to understand the automorphisms on a the subposet consisting of elements of rank at most n for each n. By inspection, it is easy to see that, up to rank 2, this is the only nontrivial automorphism, and that σ is in fact an automorphism on all of $Z(1)$. Now, suppose τ were any automorphism. Then either $\tau(11) = 11$ or $\tau(11) = 2$. In the latter case, we can compose τ with σ, so without loss of generality, we may assume that $\tau(11) = 11$. Hence, τ is the identity up to rank 2. Suppose that τ were not the identity. Then there would be some minimal rank n so that τ were not the identity up to rank n. Hence, there must be an element x of rank n so that $\tau(x) \neq x$. But since τ is the identity on rank $n-1$, it must be the case that x and $\tau(x)$ cover exactly the same elements, so by a previous discussion, x and $\tau(x)$ cover exactly one element. The elements x of rank at least 3 covering exactly one element are precisely those beginning with a 1. But if x and y are two distinct elements beginning with a 1, then they cover no common elements. Hence there are no additional automorphisms. ∎

One question we might ask is whether it is possible to start with a finite poset that "appears" to be the first few levels of an r-differential poset, and complete it into a full r-differential poset. Indeed, this is possible, and the Fibonacci posets can be constructed in this way.

Definition 8. Let P be a finite graded poset of rank n that satisfies the first two conditions for being an r-differential poset, and also satisfies the third for any element x of rank less than n. We call such a poset an r-differential poset up to rank n.

We can extend such a P to an r-differential poset ΩP up to rank $n+1$, as follows: For each $s \in P_{n-1}$, construct an element t to have rank $n+1$, so that t covers u if and only if u covers s. Then, add r new elements $x_1, \ldots, x_r$ of rank $n+1$ for each $x \in P_n$, so that x_i covers x and nothing else. We may iterate this process infinitely many times to get an r-differential poset $\Omega^\infty P$. Here is a picture of this "reflection" process, taken from [2]:



So, one way to construct the Fibonacci poset $Z(r)$ is to start with the trivial poset $\widehat{0}$, which is r-differential up to rank 0, and take $\Omega^\infty \widehat{0}$. The Young poset, however, does not arise from completing a 1-differential poset up to rank n, for any n. To see this, note that if P is the completion of an r-differential poset up to rank n, then for $i > n$, we have

$$p_i(P) = rp_{i-1}(P) + p_{i-2}(P),$$

and the partition function satisfies no such recursion. More generally, for n sufficiently large, if we truncate the Young poset up to rank n to get $Y[n]$ and look at $\Omega^\infty Y[n]$, these are all distinct. Hence, there are infinitely many pairwise nonisomorphic 1-differential posets.

2. The U and D Operators

The real interest in differential posets comes from two linear transformations associated to them. Let K be a field, and let S be a set. Let $K[S]$ denote the K-vector space with basis S, and let $K[\![S]\!]$ denote the K-vector space of all (possibly infinite) formal linear combinatorions of elements of S. For a locally finite poset P and $x \in P$, let

$$C^-(x) = \{y \in P : x \text{ covers } y\},$$
$$C^+(x) = \{y \in P : y \text{ covers } x\}.$$

Definition 9. Let P be a locally finite poset so that for all $x \in P$, $C^-(x)$ and $C^+(x)$ are finite. Define two linear maps $U, D : K[\![P]\!] \to K[\![P]\!]$ by

$$Ux = \sum_{y \in C^+(x)} y,$$
$$Dx = \sum_{y \in C^-(x)} y.$$

Note that the sums defining Ux and Dx are finite, so these transformations do in fact make sense. Furthermore, when restricted to $K[P]$, U and D are still linear transformations.

Combinatorially, Ux and Dx will be useful for counting paths in P. However, if we have a probabilistic take on life, we may prefer to take random walks and look at the behavior of such walks. For such people, it will be more natural to look at the transformations

$$\widetilde{U}x = \frac{1}{|C^+(x)|} \sum_{y \in C^+(x)} y,$$
$$\widetilde{D}x = \frac{1}{|C^-(x)|} \sum_{y \in C^-(x)} y.$$

We will briefly mention random walks on differential posets later in this paper.

There are now two easy results we can prove about these operators and differential posets. For a subset $S \subset P$, we write $\underline{S} = \sum_{s \in S} s \in K[\![P]\!]$.

Theorem 10. (1) *Let P be a locally finite graded poset with $\widehat{0}$ and having finitely many elements of each rank. Then the following are equivalent for an integer r.*
(a) *P is r-differential.*
(b)

$$DU - UD = rI. \tag{$*$}$$

(2) $D\underline{P} = (U + r)\underline{P}$.

Proof. (1) Let $x \in P$. Then $DUx = \sum_y c_y y$, where $c_y = \#(C^+(x) \cap C^+(y))$, and $UDx = \sum_y d_y y$, where $d_y = \#(C^-(x) \cap C^-(y))$, so $DU - UD = rI$ if and only if

$$\#(C^+(x) \cap C^+(y)) = \#(C^-(x) \cap C^-(y))$$

for all distinct $x, y \in P$ and $\#C^+(x) - \#C^-(x) = r$.

(2) Suppose $D\underline{P} = \sum_x a_x x$. Then $a_x = \#C^+(x)$. Likewise, if $(U + r)\underline{P} = \sum_x b_x x$, then $b_x = r + \#C^-(x)$. This is exactly the third condition for being a differential poset.

■

Proposition 11. *Let P be an r-differential poset, and let $s(n, k)$ and $S(n, k)$ be the Stirling numbers of the first and second kinds, respectively. Then*

$$(UD)^n = \sum_{k=0}^{n} r^{n-k} S(n, k) U^k D^k$$

and

$$U^n D^n = \sum_{k=0}^{n} r^{n-k} s(n, k)(UD)^k.$$

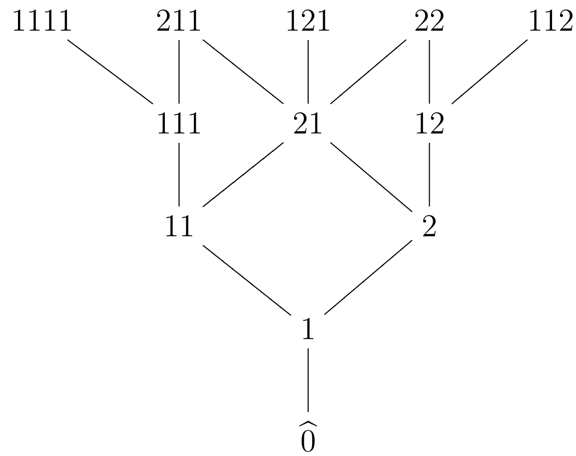
We may interpret this result as telling us about a random walk on the levels of P, as follows. For $x \in P$ of rank k, we first move to some $y \in C^-(x)$ and then move to some $z \in C^+(y)$, so that the ranks of x and z are equal. (Note that this random walk is not quite the same as moving to some $y \in C^+(x)$ and then to some $z \in C^-(y)$. For example, if $P = Z(r)$ is the Fibonacci poset, the latter walk is more likely to remain at x, since there are r elements of $C^+(x)$ which cover only x.)

The proposition allows us a path to studying the dynamics and stationary distribution of this walk: we'd like to understand the dynamics of $(UD)^n$ as n gets large, but on a fixed level k, it's equal to the *finite* sum

$$\sum_{\ell=0}^{k} r^{n-\ell} S(n, \ell) U^\ell D^\ell,$$

since if we go down at least $k + 1$ times, we go below the $\widehat{0}$ element, and hence get zero. I suspect that attempting to carry out this procedure in general will be very difficult.

Fortunately, through different means, we can work out the stationary distribution of the walk. Let's look at $Z(1)$ first before doing the general case. The beginning of the poset looks like



The transition matrix for the level four random walk is

$$\begin{pmatrix} 1/2 & 1/4 & 0 & 0 & 0 \\ 1/2 & 5/12 & 1/3 & 1/6 & 0 \\ 0 & 1/6 & 1/3 & 1/6 & 0 \\ 0 & 1/6 & 1/3 & 5/12 & 1/2 \\ 0 & 0 & 0 & 1/4 & 1/2 \end{pmatrix}.$$

Its stationary distribution is

$$\frac{1}{7}(1,2,1,2,1)^T.$$

The interpretation of this is that the stationary distribution at x is proportional to the size of $C^-(x)$. This holds more generally for all differential posets P.

To see this, we'll look at two consecutive levels $k-1$ and k of P, and we'll consider its transition matrix A. Let $\mathbf{v}$ be the vector whose x component is the degree of x. Then the x component of $A\mathbf{v}$ is

$$\sum_{z \to x} \frac{1}{\deg(z)} \deg(z) = \deg(x).$$

Hence $\mathbf{v}$ is the stationary distribution for A, and hence the stationary distribution for A^2 as well. Restricting A and $\mathbf{v}$ to those components of level k gives the desired result.

At this point, and probably much earlier, the reader may wonder why differential posets are so named. Suppose $f(x)$ is a C^∞ function, and $D = \frac{d}{dx}$. Then we have

$$D(xf(x)) = xD(f(x)) + f(x),$$

or

$$(Dx - xD)f(x) = f(x),$$

so the operator $Dx - xD$ is the identity operator. (In particular, Dx and xD fail to commute.) Replacing x by U here, we recover equation $(*)$ above, for $r = 1$. More generally, if we're working in an r-differential poset and g is any polynomial, then

$$Df(U) - f(U)D = rf'(U).$$

This is the origin of the nomenclature.

3. Path Counting

One of the primary uses for the operators U and D on a differetial poset is for counting Hasse walks on P.

Definition 12. A Hasse walk of length n on a poset P is a sequence $x_0, \ldots, x_n$ of elements of P so that for each i with $0 \le i \le n-1$, $x_{i+1} \in C^+(x_i) \cup C^-(x_i)$.

Frequently, we wish to be more specific about the walks allowed: for instance, perhaps we insist that the first step goes up, then the next two go down, and so forth. We'll frequently also specify conditions for the allowed starting and ending points of the walk.

In order to count allowable Hasse walks, we first have to define a topology on $K[\![P]\!]$. We do this by giving $K[P]$ the discrete topology, and giving $K[\![P]\!]$ the topology defined by $\sum_{i=0}^{\infty} a_i x_i$ being the limit of $\sum_{i=0}^{N} a_i x_i$.

This allows us to define a bilinear form $\langle \cdot, \cdot \rangle : K[\![P]\!] \times K[P] \to K$ by $\langle x, y \rangle = \delta_{xy}$, and extending by bilinearity, together with continuity in the first slot. Since we only allow finite linear combinations in the second slot, the sum

$$\left\langle \sum a_x x, \sum b_x x \right\rangle = \sum a_x b_x$$

is finite; we cannot extend this form to $K[\![P]\!]$ in the second slot, since we cannot interpret infinite sums in K.

If P is a differential poset, then U and D are adjoint operators, since

$$\langle Dx, y \rangle = \langle x, Uy \rangle = \begin{cases} 1 & x \in C^+(y), \\ 0 & \text{otherwise,} \end{cases}$$

and similarly $\langle Ux, y \rangle = \langle x, Dy \rangle$.

Let's also set $\alpha(0 \to n)$ equal to the number of Hasse walks $\widehat{0} = x_0 \to x_1 \to \cdots \to x_n$ on P, so that each x_{i+1} covers x_i (so in particular the level of x_i is i). Clearly,

$$\langle D^n \underline{P}, \widehat{0} \rangle = \alpha(0 \to n),$$

and the number of Hasse walks starting and ending at level n is

$$\langle (U + D)^n \underline{P}, \widehat{0} \rangle,$$

a number we'll call δ_n.

Surprisingly, $\alpha(0 \to n)$ does not depend much on the choice of P.

Theorem 13. *Let P be an r-differential poset. Then*

$$\sum_{n\geq 0}\alpha(0\to n)\frac{t^n}{n!}=\exp\left(rt+\frac{1}{2}rt^2\right).$$

Proof. See Stanley, Proposition 3.1, page 927–928. ∎

In the case where P is the Young poset Y, this theorem (together with some facts on symmetric functions) tells us that $\alpha(0\to n)$ in Y is equal to the number of $\sigma\in S_n$ so that $\sigma^2=1$. Note that $\alpha(0\to n)$ is equal to the number of standard Young tableaux of size n. In fact, there's an elegant bijective proof of this, based on the Robinson-Schensted Correspondence. Generally, the Robinson-Schensted Correspondence associates to some $\sigma\in S_n$ a pair (T_1,T_2) of standard Young tableaux of the same shape and size n. However, associated to σ^{-1} is the pair (T_2,T_1), so when σ is an involution, we get the same tableau twice. Since the correspondence is bijective, it restricts to a bijection between involutions in S_n and standard Young tableaux of size n.

Let's illustrate the Robinson-Schensted Correspondence for an involution in S_8. Consider the permutation $\sigma=(14)(26)(78)$. We first write down the permutation as 46315287, where the i^{th} place is $\sigma(i)$. We construct two tableaux out of this: in the first, we try to place each new number as it appears in the sequence above in the first row of the tableau, bumping other digits out of the way to lower rows if necessary. In the second, we track the order of the changing shape of the tableau. Hence, after one step the tableaux are

$$\begin{array}{|c|}\hline 4\\\hline\end{array}\,,\qquad \begin{array}{|c|}\hline 1\\\hline\end{array}\,.$$

After the second step, the tableaux become

$$\begin{array}{|c|c|}\hline 4&6\\\hline\end{array}\,,\qquad \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\,.$$

When we try to place the 3, we need to bump the 4 down, so after the third step, we get

$$\begin{array}{|c|c|}\hline 3&6\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array}\,,\qquad \begin{array}{|c|c|}\hline 1&2\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}\,.$$

Continuing on, we get

$$\begin{array}{|c|c|}\hline 1&6\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1} \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array}\,,\qquad \begin{array}{|c|c|}\hline 1&2\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1} \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array}\,,$$

then

$$\begin{array}{|c|c|}\hline 1&5\\\hline 3&6\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array}\,,\qquad \begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array}\,,$$

then

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline \end{array}, \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline \end{array},$$

then

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 8 \\ \hline 3 & 5 \\ \cline{1-2} 4 & 6 \\ \cline{1-2} \end{array}, \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 7 \\ \hline 3 & 5 \\ \cline{1-2} 4 & 6 \\ \cline{1-2} \end{array},$$

then

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 7 \\ \hline 3 & 5 & 8 \\ \hline 4 & 6 \\ \cline{1-2} \end{array}, \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 7 \\ \hline 3 & 5 & 8 \\ \hline 4 & 6 \\ \cline{1-2} \end{array}.$$

As we expected, since σ is an involution, the two tableaux are identical.

It will be useful to generalize the α notation to other types of Hasse walks. It is easiest to explain the notation by means of an example: $\alpha(4 \to 6 \to 2 \to 5)$ denotes the number of sequences $(x_0, \ldots, x_9)$ so that $x_0, x_1, \ldots, x_9$ have ranks $4, 5, 6, 5, 4, 3, 2, 2, 3, 4, 5$, respectively, and

$$x_0 < x_1 < x_2 > x_3 > x_4 > x_5 > x_6 < x_7 < x_8 < x_9.$$

For $x \in P$, let $e(x)$ be the number of maximal chains from $\widehat{0}$ to x. Hence, if P_n denotes the set of elements of P of rank n, then we have

$$\alpha(0 \to n \to 0) = \sum_{x \in P_n} e(x)^2, \qquad \alpha(n \to 0 \to n) = \left(\sum_{x \in P_n} e(x) \right)^2.$$

For $x \in P$, let $\rho(x)$ denote the rank of x. We let

$$F(P, q) = \sum_{x \in P} q^{\rho(x)} = \sum_{n \geq 0} \alpha(n) q^n.$$

It is well-known, for example, that

$$F(Y, q) = \prod_{n \geq 1} (1 - q^n)^{-1}.$$

4. The Characteristic Polynomial

Since differential posets come equipped with two particularly well-behaved linear transformations, it is natural to use linear algebra in order to study their properties. For example, the eigenvalues of U and D, restricted to the j^{th} level, are relevant for counting the number of elements of rank j in a poset P.

Theorem 14. *Let P be an r-differential poset, let p_n be the number of elements of rank n, and set $\Delta p_n = p_n - p_{n-1}$. Then, for $j \in \mathbb{N}$, the characteristic polynomial of UD, restricted to the rank j elements of P, is equal to*

$$\prod_{i=0}^{j} (\lambda - ri)^{\Delta p_{j-i}}.$$

On the other hand, the characteristic polynomial of DU, restricted to the rank j elements of P, is equal to

$$\prod_{i=0}^{j} (\lambda - r(i+1))^{\Delta p_{j-i}}.$$

In particular, D is surjective, and U is injective.

Proof. See [3], Theorem 4.1 and Corollary 4.2, pages 940–941. ∎

Corollary 15. *For any differential poset P, $p_0 \le p_1 \le p_2 \le \cdots$.*

Proof. The operator U is injective. ∎

It turns out that, if we look at just a small piece of a differential poset P, we can reconstruct much information about the entire poset. Let $P_{a,b}$ be the subposet of P consisting of all elements x with $a \le \rho(x) \le b$.

Theorem 16. *Suppose P and Q are r- and s-differential posets, respectively. Suppose that, for some $i, j \in \mathbb{N}$, we have $P_{i-1,i} \cong Q_{j-1,j}$ as posets (or even just as graphs). Then one of these two possibilities holds:*

(1) *$r = s$, $i = j$, and $p_k = q_k$ for $0 \le k \le j$.*
(2) *One of (r, i) and (s, j) is equal to $(1, 2)$, and the other is equal to $(2, 1)$. Also, $P_{i-1,i} \cong Q_{j-1,j} \cong K_{1,2}$.*

Proof. This is Corollary 4.5, pages 941–942 in [3] ∎

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