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Complex Analysis and Operator Theory

A Hilbert Space Approach to Bounded Analytic Interpolation

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Abstract. We generalize several results on bounded analytic interpolation of Fitzgerald and Horn, which work by majorization by positive definite kernels, to the cases of several complex variables and operator-valued interpolation. Using a lemma of Kolmogorov, we complement a simplification due to Szafraniec in the proofs of the theorems.

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1. Introduction

Throughout this paper, $\mathbb{D}$ will denote the (open) unit disc in $\mathbb{C}$, $H^\infty(\mathbb{D})$ the Hardy space of bounded analytic functions inside the disc, and if U is an open subset of $\mathbb{C}^n$, then $L^2_a(U)$ denotes the Bergman space on U.

Modern bounded analytic interpolation theory began with the works of Pick and Nevanlinna in the early part of the 20$^{\text{th}}$ century. Pick and Nevanlinna considered the following problem:

Question. *Given N points $z_1, \dots, z_N$ in the unit disc $\mathbb{D}$ and N complex numbers $w_1, \dots, w_N$, when does there exist an analytic function $f : \mathbb{D} \to \mathbb{D}$ so that $f(z_i) = w_i$ for $1 \le i \le N$?*

This question was first answered in [5] and [6], as follows:

Theorem (Pick and Nevanlinna). *There is a function f in $H^\infty(\mathbb{D})$ with $\|f\|_{H^\infty(\mathbb{D})} \le 1$ which satisfies the above interpolation criterion if and only if the matrix*

$$\left(\frac{1 - w_i\bar{w}_j}{1 - z_i\bar{z}_j}\right)_{i,j=1}^N$$

is a positive semidefinite matrix. Furthermore, f is unique if and only if the rank of the above matrix is strictly less than N.

Since then, a variety of interpolation problems has been investigated. Much of the history is described in detail by Agler and McCarthy in [1]. However, we shall focus our attention on some particular results of Fitzgerald and Horn which are not discussed in the book of Agler and McCarthy. Fitzgerald and Horn solved several interpolation problems in [2] and [3] by a repeated use of majorization by positive definite kernels. The main point of this note is to underline the existence, in many of their arguments, of a canonical Hilbert space which provides the domain of factorization of the respective kernels. Nowadays, this technique, also known as the realization theory of transfer functions, is widely used in the theory of electrical engineering.

Definition 1. If X is a set, then a positive semidefinite kernel on X is a map $\mathscr{K} : X \times X \to \mathbb{C}$ such that the inequality

$$\sum_{i,j=1}^{n} \mathscr{K}(x_i, x_j)\lambda_i\bar{\lambda}_j \geq 0$$

holds for any finite sequence $x_1, \ldots, x_N$ of points in X and any sequence of complex numbers $\lambda_1, \ldots, \lambda_N$ of the same length. A positive semidefinite kernel $\mathscr{K} : X \times X \to \mathbb{C}$ is called positive definite if $\mathscr{K}(x, x) \neq 0$ for every $x \in X$. If $\mathscr{D} \subseteq \mathbb{C}^d$ is a domain, then a sesquianalytic kernel on $\mathscr{D}$ is a map $\mathscr{K} : \mathscr{D}\times\mathscr{D} \to \mathbb{C}$ so that $\mathscr{K}(z, \zeta)$ is analytic in z and anti-analytic in ζ. If a kernel $\mathscr{K}$ satisfies $\mathscr{K}(x, y) = \overline{\mathscr{K}(y, x)}$, then $\mathscr{K}$ is said to be self-adjoint.

Note that for inequalities such as

$$\sum_{i,j=1}^{n} \mathscr{K}(x_i, x_j)\alpha_i\bar{\alpha}_j \geq 0$$

to make sense in general, it is necessary that $\mathscr{K}$ be a self-adjoint kernel, as otherwise such sums need not be real-valued.

We begin by proving a well-known lemma, due to Kolmogorov [4], which we will use as our main tool throughout this note.

Lemma 2 (Kolmogorov's Factorization Lemma). *Let X be a set and $\mathscr{K} : X \times X \to \mathbb{C}$ be a positive semidefinite kernel on X. Then there exists a Hilbert space $\mathscr{H}$ and a map $\psi : X \to \mathscr{H}$ such that $\mathscr{K}(x, y) = \langle\psi(x), \psi(y)\rangle$ for all $x, y \in X$.*

Proof. Let $\mathscr{V} = \bigoplus_{x\in X} \mathbb{C}_x$, where each $\mathbb{C}_x \cong \mathbb{C}$. Define an inner product on $\mathscr{V}$ by

$$\big\langle (v_x), (w_y) \big\rangle := \sum_{x,y} \mathscr{K}(x, y)v_x\bar{w}_y\,.$$

Since $\mathscr{K}$ is a positive semidefinite kernel, $\langle\cdot,\cdot\rangle$ is a positive semidefinite bilinear form. Let $\mathscr{N} = \{v \in \mathscr{V} : \|v\| = 0\}$. By the Cauchy-Schwarz inequality, if $x, y \in \mathscr{N}$, then $x \pm y \in \mathscr{N}$ as well, so $\mathscr{N}$ is a subspace of $\mathscr{V}$. Therefore we may consider the quotient space $\mathscr{V}/\mathscr{N}$, on which the induced inner product is positive definite. Now let $\mathscr{H} = \overline{\mathscr{V}/\mathscr{N}}$, the completion of $\mathscr{V}/\mathscr{N}$. For $x, y \in X$, set $\delta_{xx} = 1$ and $\delta_{xy} = 0$ if $x \neq y$. Then define $\psi : X \to \mathscr{H}$ by $\psi(x) = (\delta_{xy})_{y \in X} + \mathscr{N}$. □

Remark. In the above lemma, if $X = \mathscr{D}$ is a domain in $\mathbb{C}^d$ and $\mathscr{K}$ is a sesquianalytic kernel, then $\psi : \mathscr{D} \to \mathscr{H}$ is holomorphic.

We can now use this lemma to prove several results in interpolation theory.

2. Main results

Proposition 3. *Let $\mathcal{S}$ be a uniqueness set for the domain $\mathscr{D} \subseteq \mathbb{C}^d$, and let $\mathscr{K}_1$ and $\mathscr{K}_2$ be two positive definite sesquianalytic kernels on the domain $\mathscr{D}$ so that the inequality*

$$\sum_{i,j=1}^{n} \mathscr{K}_1(z_i, z_j)\alpha_i\bar{\alpha}_j \geq \sum_{i,j=1}^{n} \mathscr{K}_2(z_i, z_j)\alpha_i\bar{\alpha}_j \tag{1}$$

holds for any finite sequence of points $z_1, \ldots, z_n \in \mathcal{S}$ and any sequence of complex numbers $\alpha_1, \ldots, \alpha_n$ of the same length. Then (1) holds for all $z_1, \ldots, z_n \in \mathscr{D}$.

Proof. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces, and let $\psi_i : \mathscr{D} \to \mathscr{H}_i$ (for $i = 1, 2$) be such that $\mathscr{K}_i(z, w) = \langle \psi_i(z), \psi_i(w) \rangle$, as in Lemma 2. Define

$$T : \bigvee_{z \in \mathcal{S}} \psi_1(z) \to \bigvee_{z \in \mathcal{S}} \psi_2(z)$$

by

$$\sum_i \alpha_i \psi_1(z_i) \mapsto \sum_i \alpha_i \psi_2(z_i)\,.$$

By (1), T is well-defined and bounded, with $\|T\| \leq 1$. Now extend T to $\widetilde{T} : \mathscr{H}_1 \to \mathscr{H}_2$ in such a way that $\|\widetilde{T}\| \leq 1$ (for example, by setting $\widetilde{T}(h) = 0$ for $h \in \big(\bigvee_{z \in \mathcal{S}} \psi_1(z)\big)^{\perp}$).

Now define $\widetilde{\mathscr{K}_2} : \mathscr{D} \times \mathscr{D} \to \mathbb{C}$ by

$$\widetilde{\mathscr{K}_2}(z, w) = \big\langle \widetilde{T}\psi_1(z), \widetilde{T}\psi_1(w) \big\rangle\,.$$

Because $\widetilde{\mathscr{K}_2}\,|_{\mathcal{S} \times \mathcal{S}} = \mathscr{K}_2$, we may replace $\mathscr{K}_2$ by $\widetilde{\mathscr{K}_2}\,|_{\mathcal{S} \times \mathcal{S}}$ in (1). With $\widetilde{\mathscr{K}_2}$ in place of $\mathscr{K}_2$, (1) holds on all of $\mathscr{D}$ because of the contractive property of $\widetilde{T}$. But $\mathscr{K}_2$ and $\widetilde{\mathscr{K}_2}$ agree on the uniqueness set $\mathcal{S}$, so they must agree on all of $\mathscr{D}$. □

Sometimes we are given a kernel $\mathscr{K}$ on a domain $\mathscr{D}$ which is known to be positive semidefinite on a subdomain $\mathscr{D}'$. We may wish to know under which conditions $\mathscr{K}$ is actually positive semidefinite on all of $\mathscr{D}$. We isolate below a condition that guarantees positive semidefiniteness on the full domain.

Theorem 4. *Let $\mathscr{D} \subseteq \mathbb{C}^d$ be a connected open domain, let $\mathscr{D}' \Subset \mathscr{D}$ be a relatively compact subdomain, and let $\mathcal{S}$ be a uniqueness set for $\mathscr{D}'$. Let $\mathscr{K} : \mathscr{D} \times \mathscr{D} \to \mathbb{C}$ be a sesquianalytic kernel which is positive semidefinite on $\mathcal{S} \times \mathcal{S}$. Then $\mathscr{K}$ is positive semidefinite on $\mathscr{D} \times \mathscr{D}$.*

Proof. Let $\mathscr{D}' \subseteq U \Subset \mathscr{D}$. We claim that $\mathcal{S}$ is a uniqueness set for the Bergman space $L^2_a(U)$. To see this, take $f \in L^2_a(U)$. Then $f \mid_{\mathscr{D}'}$ is analytic, and so if $f \mid_{\mathcal{S}} = 0$, then $f \mid_{\mathscr{D}'} = 0$. But $\mathscr{D}'$ is open in U, so $f = 0$ on U.

Let $B : U \times U \to \mathbb{C}$ be the Bergman kernel on U. Then, for each $s \in \mathcal{S}$, the function $B(\cdot, s)$ is in $L^2_a(U)$. Let $W = \bigvee_{s \in \mathcal{S}} B(\cdot, s)$. We claim that W is dense in $L^2_a(U)$. To see this, let $f \in L^2_a(U)$, and suppose $f \in W^\perp$. Then

$$0 = \langle f, B(\cdot, s) \rangle = \int_U f(\zeta) \overline{B(\zeta, s)}\, d\zeta = f(s)$$

for each $s \in \mathcal{S}$. Hence $f \equiv 0$, and so W is indeed dense in $L^2_a(U)$.

Now pick a function $\varphi \in W$, given by

$$\varphi(u) = \sum_{i=1}^{n} \alpha_i B(u, s_i)\,,$$

where each $s_i \in \mathcal{S}$ and $\alpha_i \in \mathbb{C}$. Since $\mathscr{K}$ is positive semidefinite on $\mathcal{S} \times \mathcal{S}$, we have

$$\int_{U \times U} \varphi(u) \bar{\varphi}(v) \mathscr{K}(u, v)\, du\, dv \geq 0 \tag{2}$$

for any $\varphi \in W$. Since W is dense in $L^2_a(U)$, any function in $L^2_a(U)$ must also have property (2) as well. In particular, (2) holds for all functions of the form $\theta(u) = \sum_{i=1}^n \alpha_i B(u, z_i)$, where $z_1, \ldots, z_n \in U$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. But any finite sequence of points in $\mathscr{D}$ lies in some such U. Hence $\mathscr{K}$ is positive semidefinite on $\mathscr{D} \times \mathscr{D}$, as desired. □

The following two theorems are modest generalizations of results by Fitzgerald and Horn in [2] and [3]. We again emphasize the novel proof technique making use of Lemma 2.

Theorem 5 (Fitzgerald and Horn). *Let $\mathscr{D} \subseteq \mathbb{C}^d$ be a domain, and let $\mathscr{K}(z, \zeta)$ be a positive semidefinite sesquianalytic kernel on $\mathscr{D}$. Let P be a set of points in $\mathscr{D}$. Suppose $\{w_p\}_{p \in P}$ is a set of complex numbers. If*

$$\sum_{n,m=1}^{N} \gamma_n \bar{\gamma}_m \mathscr{K}(z_n, z_m) \geq \left| \sum_{n=1}^{N} \gamma_n w_{z_n} \right|^2 \tag{3}$$

holds for any finite sequence $z_1, \ldots, z_N \in P$ and for any sequence of complex numbers $\gamma_1, \ldots, \gamma_N$ of the same length, then there exists an analytic function $v : \mathscr{D} \to \mathbb{C}$ such that $v(p) = w_p$ for all $p \in P$. Moreover, the inequality

$$\sum_{n,m=1}^{N} \gamma_n \bar{\gamma}_m \mathscr{K}(z_n, z_m) \geq \left| \sum_{n=1}^{N} \gamma_n v(z_n) \right|^2 \tag{4}$$

holds for any finite sequence $z_1, \ldots, z_N \in \mathscr{D}$ and for any sequence of complex numbers $\gamma_1, \ldots, \gamma_N$ of the same length.

Proof. By Lemma 2, there is a Hilbert space $\mathscr{H}$ and an analytic function $\psi : \mathscr{D} \to \mathscr{H}$ such that $\mathscr{K}(z, \zeta) = \langle \psi(z), \psi(\zeta) \rangle$. Then equation (3) is equivalent to

$$\left\| \sum_{i=1}^{N} \gamma_i \psi(z_i) \right\|^2 \geq \left| \sum_{j=1}^{N} \gamma_j w_{z_j} \right|^2 .$$

Define a linear map

$$L : \bigvee_{p \in P} \psi(p) \to \mathbb{C}$$

by

$$L\left(\sum_{i=1}^{n} \gamma_i \psi(z_i) \right) = \sum_{i=1}^{n} \gamma_i w_{z_i} .$$

We can extend L to $\widetilde{L} : \mathscr{H} \to \mathbb{C}$ by setting $\widetilde{L}(h) = 0$ for $h \in \left(\bigvee_{p \in P} \psi(p) \right)^{\perp}$. Then $\widetilde{L}$ is a bounded linear functional with $\|\widetilde{L}\| = \|L\| \leq 1$. Then by the Riesz representation theorem, $\widetilde{L}(h) = \langle h, \xi \rangle$ for some $\xi \in \mathscr{H}$. Hence $\widetilde{L}$ is analytic in the variable h. Finally, set $v(z) = \langle \psi(z), \xi \rangle$. That the inequality (4) holds for any finite sequence of points in $\mathscr{D}$ follows because $\|\widetilde{L}\| \leq 1$. □

The result can be extended to include the case of interpolation in several $\mathbb{C}^d$ variables without making great changes. In the following, we show how this can be done in the case of two $\mathbb{C}^d$ variables.

Theorem 6 (Fitzgerald and Horn). *Let $\mathscr{D}_1, \mathscr{D}_2 \subseteq \mathbb{C}^d$ be domains, let $\mathscr{K}_1(z, \zeta)$ be a positive semidefinite sesquianalytic kernel on $\mathscr{D}_1$, and let $\mathscr{K}_2(z, \zeta)$ a positive semidefinite sesquianalytic kernel on $\mathscr{D}_2$. Let P be a set of points in $\mathscr{D}_1$ and Q a set of points in $\mathscr{D}_2$. Suppose $\{w_{p,q}\}_{p \in P, q \in Q}$ is a doubly indexed set of complex numbers. If*

$$\left[\sum_{n,m=1}^{N_1} \gamma_n \bar{\gamma}_m \mathscr{K}_1(z_n, z_m) \right] \left[\sum_{n,m=1}^{N_2} \delta_n \bar{\delta}_m \mathscr{K}_2(\zeta_n, \zeta_m) \right] \geq \left| \sum_{n,m=1}^{N_1, N_2} \gamma_n \delta_m w_{z_n, \zeta_m} \right|^2 \tag{5}$$

holds for any pair of finite sequences $z_1, \ldots, z_{N_1} \in P$ and $\zeta_1, \ldots, \zeta_{N_2} \in Q$ and for any pair of sequences of complex numbers $\gamma_1, \ldots, \gamma_{N_1}$, $\delta_1, \ldots, \delta_{N_2}$ of the same lengths, then there exists $v(z, \zeta)$ on $\mathscr{D}_1 \times \mathscr{D}_2$ analytic in z and ζ such that $v(p, q) = w_{p,q}$ for all $p \in P$, $q \in Q$. Moreover, the inequality

$$\left[\sum_{n,m=1}^{N_1} \gamma_n \bar{\gamma}_m \mathscr{K}_1(z_n, z_m) \right] \left[\sum_{n,m=1}^{N_2} \delta_n \bar{\delta}_m \mathscr{K}_2(\zeta_n, \zeta_m) \right] \geq \left| \sum_{n,m=1}^{N_1, N_2} \gamma_n \delta_m v(z_n, \zeta_m) \right|^2 \tag{6}$$

holds for any pair of finite sequence $z_1, \ldots, z_{N_1} \in \mathscr{D}_1$ and $\zeta_1, \ldots, \zeta_{N_2} \in \mathscr{D}_2$ and for any pair of sequences of complex numbers $\gamma_1, \ldots, \gamma_{N_1}$, $\delta_1, \ldots, \delta_{N_2}$ of the same lengths.

Proof. By Lemma 2, there are Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ and analytic functions $\psi : \mathscr{D}_1 \to \mathscr{H}_1$, $\varphi : \mathscr{D}_2 \to \mathscr{H}_2$ such that $\mathscr{K}_1(z,\zeta) = \langle \psi(z), \psi(\zeta)\rangle_1$ and $\mathscr{K}_2(z,\zeta) = \langle \varphi(z), \varphi(\zeta)\rangle_2$. Then equation (5) is equivalent to

$$\left\|\sum_{i=1}^{N_1} \gamma_i \psi(z_i)\right\|^2 \left\|\sum_{i=1}^{N_2} \delta_i \varphi(\zeta_i)\right\|^2 \geq \left|\sum_{n,m=1}^{N_1,N_2} \gamma_n \delta_m w_{z_n,\zeta_m}\right|^2 .$$

Define a bilinear function

$$L : \bigvee_{p\in P} \psi(p) \times \bigvee_{q\in Q} \varphi(q) \to \mathbb{C}$$

by

$$L\left(\sum \gamma_i \psi(z_i), \sum \delta_i \varphi(\zeta_i)\right) = \sum \gamma_i \delta_j w_{z_i,\zeta_j} .$$

We can extend L to $\widehat{L} : \mathscr{H}_1 \times \bigvee_{q\in Q} \varphi(q) \to \mathbb{C}$ by setting

$$\widehat{L}\left(h, \sum \delta_i \varphi(q)\right) = 0$$

for $h \in \left(\bigvee_{p\in P} \psi(p)\right)^{\perp}$, and then we can extend $\widehat{L}$ to $\widetilde{L} : \mathscr{H}_1 \times \mathscr{H}_2 \to \mathbb{C}$ by setting

$$\widetilde{L}(h_1, h_2) = 0$$

for $h_2 \in \left(\bigvee_{q\in Q} \varphi(q)\right)^{\perp}$. Then $\widetilde{L} : \mathscr{H} \to \mathbb{C}$ is bilinear and bounded with $\|\widetilde{L}\| = \|L\| \leq 1$. If we fix h_2, then by the Riesz representation theorem we have $\widetilde{L}(h_1,h_2) = \langle h_1, \xi_{h_2}\rangle_1$ for some $\xi_{h_2} \in \mathscr{H}_1$. Hence $\widetilde{L}$ is analytic in the first variable. We can also fix h_1, and by the Riesz representation theorem we have $\widetilde{L}(h_1,h_2) = \langle \rho_{h_1}, h_2\rangle_2$ for some $\rho_{h_1} \in \mathscr{H}_2$. Similarly, $\widetilde{L}$ is analytic in the second variable. Finally, set $v(z,\zeta) = \langle \psi(z), \xi_{\varphi(\zeta)}\rangle_1 = \langle \rho_{\psi(z)}, \varphi(\zeta)\rangle_2$. That the inequality (6) holds for any pair of finite sequences of points in $\mathscr{D}_1$ and $\mathscr{D}_2$ follows because $\|\widetilde{L}\| \leq 1$. □

Remarks. There are many possible generalizations of these theorems. We present a few here that will be necessary to prove our next theorem:

1. The proof holds *mutatis mutandis* for more than two variables.
2. The result also holds if we replace $\mathscr{D}_2$ (for instance) by a discrete set, although the notion of analyticity is lost in this case.

Next, we extend the ideas used so far to the problem of bounded operator valued analytic interpolation.

Theorem 7. *Let $\mathscr{D} \subseteq \mathbb{C}$ be a domain, and let $\mathscr{K}(z,\zeta)$ be a positive semidefinite sesquianalytic kernel on $\mathscr{D}$. Let P be a set of points in $\mathscr{D}$. Suppose $\{T_p\}_{p\in P}$ is a set of bounded linear operators on $\ell^2(\mathbb{N})$. If*

$$\sum_{n,m=1}^{N} \gamma_n \bar{\gamma}_m \mathscr{K}(z_n, z_m) \geq \left\|\sum_{n=1}^{N} \gamma_n T_{z_n}\right\|_{op}^2$$

(*where the subscript "op" denotes the operator norm*) *holds for any finite sequence* $z_1, \ldots, z_N \in P$ *and any sequence of complex numbers* $\gamma_1, \ldots, \gamma_N$ *of the same length, then there exists an analytic function* $f : \mathscr{D} \to \mathcal{L}(\ell^2(\mathbb{N}))$ *such that* $f(p) = T_p$ *for all* $p \in P$.

Proof. Let $u = (u_1, \ldots, u_{N_1}, 0 \ldots)$ and $v = (v_1, \ldots, v_{N_2}, 0, \ldots)$ be two finitely supported sequences in $\ell^2(\mathbb{N})$. Then, for any finite sequences $z_1, \ldots, z_{N_3} \in P$ and $\gamma_1, \ldots, \gamma_{N_3} \in \mathbb{C}$ of the same length, we have

$$\|u\|^2 \|v\|^2 \sum_{n,m=1}^{N_3} \gamma_n \bar{\gamma}_m \mathscr{K}(x_n, x_m) \geq \|u\|^2 \left\| \sum_{n=1}^{N_3} \gamma_n T_{z_n} \right\|_{op}^2 \|v\|^2$$
$$\geq \left| \left\langle \sum_{n=1}^{N_3} \gamma_n T_{z_n} u, \bar{v} \right\rangle \right|^2 .$$

Now, realizing that

$$\left\langle \sum_{n=1}^{N_3} \gamma_n T_{z_n} u, \bar{v} \right\rangle = \sum_{i,j,n=1}^{N_1, N_2, N_3} u_i v_j \langle T_{z_n} e_i, e_j \rangle$$

and interpreting $\|u\|$ as

$$\|u\| = \sum_{i=1}^{N_1} u_i \bar{u}_j \delta_{ij} ,$$

where δ_{ij} is the Kronecker δ kernel on the discrete set $\mathbb{N}$ (and similarly for v), we apply the generalization of Theorem 6 to find an interpolating function $f : \mathbb{N} \times \mathbb{N} \times \mathscr{D} \to \mathbb{C}$ having $f(i, j, p) = \langle T_p e_i, e_j \rangle$ for every $p \in P$. Clearly, for each $z \in \mathscr{D}$, f defines a linear operator on $\ell^2(\mathbb{N})$ by

$$Fz = \sum_{i,j=1}^{\infty} f(i, j, z)\, e_i \otimes e_j ,$$

and F, by Theorem 6, must satisfy

$$\|u\|^2 \|v\|^2 \mathscr{K}(z, z) \geq \left| \langle Fz(u), \bar{v} \rangle \right|^2 \tag{7}$$

for any finitely supported $u, v \in \ell^2(\mathbb{N})$. Moreover, as finitely supported sequences are dense in $\ell^2(\mathbb{N})$, inequality (7) holds for arbitrary $u, v \in \ell^2(\mathbb{N})$, thus showing that

$$\|Fz\|_{op}^2 \leq \mathscr{K}(z, z) .$$

Furthermore, F is weakly analytic (and therefore analytic) because each $f(i, j, \cdot)$ is analytic. □

Fitzgerald and Horn in [2] and [3] use these results to derive various other interpolation results. The interested reader may wish to refer to the original papers.

Another theorem that Fitzgerald and Horn prove in [3] is the following. In the next statement, we simplify their proof and, at the same time, extend the result to higher dimensions.

Theorem 8. *Let $n \geq 1$ be an integer. Let $\mathscr{D} \subseteq \mathbb{C}^n$ be a domain, and let $\mathscr{K}(z,\zeta)$ be a sesquianalytic kernel on $\mathscr{D}$. Suppose that $\mathcal{S}$ is a dense subset of $\mathscr{D}$ and that $f : \mathcal{S} \to \mathbb{C}$ satisfies*

$$\sum_{i,j=1}^{3} \alpha_i \bar{\alpha}_j \mathscr{K}(z_i, z_j) \geq \left| \sum_{i=1}^{3} \alpha_i f(z_i) \right|^2 \tag{8}$$

for all $z_1, z_2, z_3 \in \mathcal{S}$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ with $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Then there is a (unique) continuous function $F : \mathscr{D} \to \mathbb{C}$ so that $F|_{\mathcal{S}} = f$. Furthermore, F is analytic on $\mathscr{D}$.

Proof. Since $\mathcal{S}$ is dense in $\mathscr{D}$, there can only be at most one such continuous function F extending f. We now show existence of a continuous function. Let $x \in \mathscr{D}$ be arbitrary, and let $\{x_n\}$ be a sequence in $\mathcal{S}$ converging to x. Let $\varepsilon > 0$. Since $\{x_n\}$ is a Cauchy sequence, there exists an integer N_ε so that whenever $n, m > N_\varepsilon$,

$$|\mathscr{K}(x_n, x_n) - \mathscr{K}(x_m, x_n)| < \frac{\varepsilon}{2}. \tag{9}$$

We now apply (8) with $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = 0$, $z_1 = x_n$, and $z_2 = x_m$. Then by (9), we have

$$|f(x_n) - f(x_m)| \leq |\mathscr{K}(x_n, x_n) - \mathscr{K}(x_m, x_n) - \mathscr{K}(x_n, x_m) + \mathscr{K}(x_m, x_m)| < \varepsilon$$

whenever $n, m > N_\varepsilon$. Therefore $\{f(x_n)\}$ is a Cauchy sequence, so we may define

$$F(x) = \lim_{n\to\infty} f(x_n),$$

which is easily checked to be well-defined. Since F is continuous on a dense subset of $\mathcal{S}$ and is defined by limiting values elsewhere, F is indeed continuous on all of $\mathscr{D}$.

We now show that the extension function F is in fact analytic on $\mathscr{D}$ if $n = 1$. Fix $z_1 \in \mathscr{D}$; we must show that F is differentiable at z_1. To do this, it suffices to show that $(F(z_1) - F(z_2))/(z_1 - z_2)$ tends to a unique finite limit as $|z_1 - z_2|$ tends to zero. It suffices to show that

$$\left| \frac{F(z_1) - F(z_2)}{z_1 - z_2} - \frac{F(z_1) - F(z_3)}{z_1 - z_3} \right|$$

tends uniformly to zero as $|z_2 - z_1|$ and $|z_3 - z_1|$ tend to zero. We can rewrite this last expression as

$$\begin{aligned} &\left| \left(\frac{1}{z_1 - z_2} - \frac{1}{z_1 - z_3} \right) F(z_1) + \frac{-1}{z_1 - z_2} F(z_2) + \frac{1}{z_1 - z_3} F(z_3) \right| \\ &\qquad = \left| \frac{z_2 - z_3}{(z_1 - z_2)(z_1 - z_3)} F(z_1) + \frac{-1}{z_1 - z_2} F(z_2) + \frac{1}{z_1 - z_3} F(z_3) \right|. \end{aligned} \tag{10}$$

We can set

$$\alpha_1 = \frac{z_2 - z_3}{(z_1 - z_2)(z_1 - z_3)}, \qquad \alpha_2 = \frac{-1}{z_1 - z_2}, \qquad \alpha_3 = \frac{1}{z_1 - z_3}.$$

Then application of the triangle inequality and equation (8) (and the fact that $\mathscr{K}$ is sesquianalytic) imply that

$$\left| \frac{F(z_1) - F(z_2)}{z_1 - z_2} - \frac{F(z_1) - F(z_3)}{z_1 - z_3} \right| \to 0$$

as z_2 and z_3 tend to zero. Hence F is analytic in the case of $n = 1$.

Now suppose $n > 1$. If we replace f by F, then (8) holds for all $z_1, z_2, z_3 \in \mathscr{D}$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ with $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Let $z_1 = (z_1^{(1)}, \ldots, z_1^{(n)})$ be an arbitrary point of $\mathscr{D}$. To show that F is analytic at z_1, fix $z_1^{(1)}, \ldots, z_1^{(i-1)}, z_1^{(i+1)}, \ldots, z_1^{(n)}$. Let $\mathscr{D}_{z_1}^{(i)}$ be the connected component of $\{w \in \mathbb{C} : (z_1^{(1)}, \ldots, z_1^{(i-1)}, w, z_1^{(i+1)}, \ldots, z_1^{(n)}) \in \mathscr{D}\}$ containing z_1. Then $\mathscr{D}_{z_1}^{(i)}$ is a domain in $\mathbb{C}$. Since (8) holds for F on $\mathscr{D}_{z_1}^{(i)}$, F is analytic at z_1 in the i^{th} variable. Then by Hartogs's Theorem (see page 28 of [9]), F is analytic at z_1 in all variables. Hence F is analytic on all of $\mathscr{D}$, as desired. □

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References

[1] J. Agler and J. E. McCarthy. *Pick Interpolation and Hilbert Function Spaces*. Graduate Studies in Mathematics **44**, American Mathematical Society, 2002.

[2] C. H. Fitzgerald. Quadratic Inequalities and Analytic Continuation. *Journal d'Analyse Mathématique*, Vol. 31 (1977): 19–46.

[3] C. H. Fitzgerald and R. A. Horn. On Quadratic and Bilinear Forms in Function Theory. *Proceedings of the London Mathematical Society*, Vol. 44, No. 3 (1982): 554–576.

[4] A. N. Kolmogorov. Stationary Sequences in Hilbert's Space. *Bulletin of the Moscow State University*, Vol. 2, No. 6 (1941): 1–40.

[5] R. Nevanlinna. Über beschränkte Funktionen, die in gegebenen Punkten vorgeschrieben Werte annehmen. *Annales Academiae Scientiarum Fennicae*, Series A, Vol. 15 (1919).

[6] G. Pick. Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden. *Mathematische Annalen*, Vol. 77 (1916): 7–23.

[7] R. M. Range. *Holomorphic Functions and Integral Representations in Several Complex Variables*. Graduate Texts in Mathematics **108**, Springer-Verlag, 1998.

[8] P. H. Szafraniec. Interpolation and Domination by Positive Definite Kernels. in *Complex Analysis – Fifth Rumanian–Finish Seminar. Part 2*. Lecture Notes in Mathematics **1014**, Springer, 1983: 291–295.

[9] J. L. Taylor. *Several Complex Variables with Connections to Algebraic Geometry and Lie Groups*. Graduate Studies in Mathematics **46**, American Mathematical Society, 2002.

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